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ON (C, α)-CONVERGENT INTEGRALS AND THEIR APPLICATION TO MATHEMATICAL PHYSICS

BY

P. K. GHOSH

(*Received January 20, 1948*)

INTRODUCTION

This paper deals with a generalisation of $(C, 1)$ convergent integrals treated by the author in a previous paper (Ghosh, 1947). As before we establish the properties of continuity, differentiability and integrability of such integrals with respect to a parameter. This makes them fit for use as solutions of differential equations with respect to the parameter or parameters and hence applicable for solution of such equations of mathematical physics.

MATHEMATICAL THEORY

We know that if a function $f(x)$ integrable $-R$ tends to l as x tends to infinity, then

$$\frac{\alpha}{x}\int_0^x \left(1-\frac{t}{x}\right)^{\alpha-1} f(t)dt \quad (\alpha > 0)$$

tends to the same limit as x tends to infinity. If $f(x)$ diverges definitely, then

$$\frac{\alpha}{x}\int_0^x \left(1-\frac{t}{x}\right)^{\alpha-1} f(t)dt$$

also diverges definitely.

Definition. *If*

$$\frac{\alpha}{x}\int_0^x \left(1-\frac{t}{x}\right)^{\alpha-1} f(t)dt$$

tends to a limit l as $x \to \infty$, we call this limit the (C, α)-limit of $f(x)$ as $x \to \infty$. [*cf.* Doetsch (1920)].

Example: In the ordinary sense $x \sin x$ oscillates infinitely as $x \to \infty$, but $(C, 2)$-limit of $x \sin x$ as $x \to \infty$, *viz.*,

$$(C, 2)\text{-}\lim_{x\to\infty} x \sin x = \lim_{x\to\infty} \frac{2}{x}\int_0^x \left(1-\frac{t}{x}\right) t \sin t dt = 0$$

In case the function $f(x)$ is not defined, say from 0 to b, we consider another function $f_1(x)$ such that

$$f_1(x) = 0, \qquad 0 \leq x < b$$
$$f_1(x) = f(x), \quad x \geq b$$

and take the limit of

$$\frac{\alpha}{x}\int_0^x \left(1-\frac{t}{x}\right)^{\alpha-1} f_1(t)dt$$

(provided the limit exists) as x tends to ∞ as the (C, α)-limit of $f(x)$ as $x \to \infty$.

Consistency Theorem. It has been established that *if $f(x)$ is (C, α)-convergent, then it is also (C, β)-convergent to the same value if $\beta > \alpha$.* If *$f(x)$ diverges definitely, then also (C, α)-limit of $f(x)$ diverges definitely.* (Doetsch, 1920, 1921).

We now consider the integral

$$F(x) = \int_c^x f(x)dx \tag{1}$$

If the (C, α)-limit of $F(x)$ as $x \to \infty$ exists, then we say that the integral on the right of (1) is (C, α)-convergent and we take this limit as the value of the infinite integral. We denote this integral by

$$(C, \alpha)\text{-}\int_c^\infty f(x)dx.$$

According to this notation

$$\int_c^\infty f(x)dx$$

of my previous paper becomes

$$(C, 1)\text{-}\int_c^\infty f(x)dx.$$

Example: Let us consider the integral

$$F(x) = \int_0^x x\sin x dx = -x\cos x + \sin x$$

In the ordinary sense $F(x)$ oscillates as $x \to \infty$, but

$$(C, 2)\text{-}\int_0^\infty x\sin x dx = 0$$

Uniform convergence

These infinite integrals may also contain parameters. Let

$$F(x, a) = \frac{\alpha}{x}\int_0^x \left(1-\frac{t}{x}\right)^{\alpha-1} \int_c^t f(t, a)dt. \tag{2}$$

If now $F(x, a)$ converges uniformly in an interval $A \leq a \leq B$ as $x \to \infty$, we say that

$$(C, \alpha)\text{-}\int_c^\infty f(x, a)dx \tag{3}$$

converges uniformly in the interval $A \leq a \leq B$.

We now proceed to study some important properties of (C, α)-convergent integrals containing a parameter.

CONTINUITY WITH RESPECT TO A PARAMETER

Theorem 1. *If $f(x, a)$ is continuous in the region $c \leq x \leq \infty$, $A \leq a \leq B$ and if the integral*

$$(C, \alpha)\text{-}\int_c^{\infty} f(x, a)dx$$

converges uniformly in the interval $A \leq a \leq B$, then the integral defines a function

$$\varphi(a) = (C, \alpha)\text{-}\int_c^{\infty} f(x, a)dx$$

continuous in that interval.

Proof. We take any point a_0 in the interval $A \leq a \leq B$. As the integral is uniformly convergent,

$$\varphi(a_0) = \frac{\alpha}{x'}\int_0^{x'} dx\left(1-\frac{x}{x'}\right)^{\alpha-1}\int_c^{x} f(t, a)dt + R_{x'}(a_0)$$

and

$$\varphi(a_0+h) = \frac{\alpha}{x'}\int_0^{x'} dx\left(1-\frac{x}{x'}\right)^{\alpha-1}\int_c^{x} f(t, a+h)dt + R_{x'}(a_0+h)$$

where both $|R_{x'}(a_0)| < \varepsilon$ and $|R_{x'}(a_0+h)| < \varepsilon$ when $x' > X(\varepsilon)$

$$\varphi(a_0+h)-\varphi(a_0) = \frac{\alpha}{x'}\int_0^{x'} dx\left(1-\frac{x}{x'}\right)^{\alpha-1}\int_c^{x}[f(t, a+h)-f(t, a)]dt + R_{x'}(a_0+h) - R_{x'}(a_0)$$

Hence

$$|\varphi(a_0+h)-\phi(a_0)| \leq \left|\frac{\alpha}{x'}\int_0^{x'} dx\left(1-\frac{x}{x'}\right)^{\alpha-1}\int_c^{x}[f(t, a_0+h)-f(t, a_0)]dt\right| + |R_{x'}(a_0+h)| + |R_{x'}(a_0)|$$

Now we keep x' fixed. Then, since

$$\frac{\alpha}{x'}\int_0^{x'} dx\left(1-\frac{x}{x'}\right)^{\alpha-1}\int_c^{x} f(t, a)dt$$

a continuous function of a in the interval $A \leq a \leq B$, it follows that

$$\left|\frac{\alpha}{x'}\int_0^{x'} dx\left(1-\frac{x}{x'}\right)\int_c^{x}[f(t, a_0+h)-f(t, a_0)]dt\right| < \varepsilon, \quad |h| < \delta.$$

Hence

$$|\varphi(a_0+h)-\varphi(a_0)| < 3\varepsilon, \quad |h| < \delta.$$

which proves the proposition.

INTEGRATION WITH RESPECT TO A PARAMETER: REVERSAL OF THE ORDER OF INTEGRATON

Theorem 2. *If the function $f(x, a)$ is continuous in the region $c \leq x < \infty$, $A \leq a \leq B$, and if the integral*

$$(C, \alpha)\text{-}\int_c^\infty f(x, a)\,dx$$

converges uniformfy in the interval $A \leq a \leq B$:

$$\phi(a) = (C, \alpha)\text{-}\int_c^\infty f(x, a)\,dx$$

then

$$\int_{a_0}^{a_1} \varphi(a)\,da = (C, \alpha)\text{-}\int_c^\infty dx \int_{a_0}^{a_1} f(x, a)\,dx$$

or

$$\int_{a_0}^{a_1} da.(C, \alpha)\text{-}\int_c^\infty f(x, a)\,dx = (C, \alpha)\text{-}\int_c^\infty dx \int_{a_0}^{a_1} f(x, a)\,dx.$$

Proof. The function $\varphi(a)$ is continuous by Theorem 1 and therefore integrable. Because of uniform convergence

$$\varphi(a) = \frac{\alpha}{x'} \int_0^{x'} dx\left(1 - \frac{x}{x'}\right)^{\alpha-1} \int_c^x f(t, a)\,dt + R_{x'}(a)$$

where $|R_{x'}(a)| < \varepsilon$ when $x' > X(\varepsilon)$.

$$\int_{a_0}^{a_1} \varphi(a)\,da = \frac{\alpha}{x'} \int_{a_0}^{a_1} da \int_0^{x'} dx\left(1 - \frac{x}{x'}\right)^{\alpha-1} \int_c^x f(t, a)\,dt + \int_{a_0}^{a_1} R_{x'}(a)\,da$$

We keep x' fixed. The order of integration in the first integral on the right can be reversed, so that

$$\int_{a_0}^{a_1} da.(C, \alpha)\text{-}\int_c^\infty f(x, a)\,dx - \frac{\alpha}{x'} \int_0^{x'} dx\left(1 - \frac{x}{x'}\right)^{\alpha-1} \int_c^x dt \int_{a_0}^{a_1} f(t, a)\,da = \int_{a_0}^{a_1} R_{x'}(a)\,da$$

Hence

$$\left| \int_{a_0}^{a_1} da.(C, \alpha)\text{-}\int_c^\infty f(x, a)\,dx - \frac{\alpha}{x'} \int_0^{x'} dx\left(1 - \frac{x}{x'}\right)^{\alpha-1} \int_c^x dt \int_{a_0}^{a_1} f(t, a)\,da \right| < \varepsilon(a_1 - a_0)$$

for $x' > X(\varepsilon)$. Therefore

$$\int_{a_0}^{a_1} da.(C, \alpha)\text{-}\int_c^\infty f(x, a)\,dx = (C, \alpha)\text{-}\int_c^\infty dx \int_{a_0}^{a_1} f(x, a)\,da.$$

DIFFERENTIATION WITH RESPECT TO A PARAMETER

Theorem 3. *Let the integral*

$$(C, \alpha)\text{-}\int_0^\infty f(x, a)\,dx$$

converge in the interval $A \leq a \leq B$:

$$\varphi(a) = (C,\ \alpha)\text{-}\int_c^{\infty} f(x,a)dx.$$

Let $\partial f/\partial a$ *exist and let the function*

$$\partial f/\partial a = f_a(x,\ a)$$

be continuous in the region $c \leq x < \infty,\ A \leq a \leq B$. *Let the integral*

$$(C,\ \alpha)\text{-}\int_c^{\infty} f_a(x,\ a)dx$$

converge uniformly in the interval $A \leq a \leq B$.

Then the function $\varphi(a)$ *has a derivative, the derivative is continuous, and is given by the last integral:*

$$\frac{d\varphi}{da} = (C,\ \alpha)\text{-}\int_c^{\infty} f_a(x,\ a)dx.$$

Proof. Let

$$\psi(a) = (C,\ \alpha)\text{-}\int_c^{\infty} f_a(x,\ a)dx$$

Then $\psi(a)$ is continuous and therefore integrable. By Theorem 2

$$\int_A^a \psi(a)da = (C,\ \alpha)\text{-}\int_c^{\infty} dx \int_A^a \frac{\partial f}{\partial a}da = (C,\ \alpha)\text{-}\int_c^{\infty} \{f(x,\ a)-f(x,\ A)\}dx$$

$$= (C,\ \alpha)\text{-}\int_c^{\infty} f(x,\ a)dx-(C,\ \alpha)\text{-}\int_c^{\infty} f(x,\ A)dx.$$

Hence

$$\int_A^a \psi(a)da = \varphi(a)-\varphi(A).$$

From the continuity of $\psi(a)$, it at once follows that

$$\psi(a) = d\varphi/da.$$

The three theorems established above for only one parameter can at once be generalised for any number of parameters.

We now proceed to generalise the concept of the volume integral:

$$\int f(x_1,\ x_2,\ x_3)dx_1dx_2dx_3 \tag{4}$$

taken over entire space. We first integrate over a sphere of radius r, the centre being at a fixed point, so that the integral is now a function of r only, *viz.*,

$$\int_{\text{sph}} f(x_1,\ x_2,\ x_3)dx_1dx_2dx_3 = F(r) \tag{5}$$

If now the $(C,\ \alpha)$-limit of $F(r)$ as $r \to \infty$ exists we say that the integral exists and its value is

$$(C,\ \alpha)\text{-}\int f(x_1,\ x_2,\ x_3)dx_1dx_2dx_3 = (C,\ \alpha)\text{-}\lim_{r\to\infty} F(r) \tag{6}$$

Here as the integral is semi-convergent its value depends on the choice of the centre of the sphere of integration and this has to be adjusted by other conditions of the problem.

We can similarly generalise the concept when the integrand contains parameters:

$$\int f(a_1, a_2, \ldots a_r, x_1, x_2, x_3)dx_1dx_2dx_3 \tag{7}$$

taken over all space, as has been done in my previous paper. In this case also the arbitrariness due to the semi-convergence of the integral remains—the position of the centre of the sphere of integration has got to be adjusted. In this case our ultimate object being solution of differential equations, we see that it can be adjusted with the help of the boundary conditions.

APPLICATIONS

For the purpose of application we have to generalise the quantum mechanical rules for the calculation of the matrix elements of an operator A, *viz.*,

$$A_{mn} = (C, \alpha)\text{-}\int \psi_m^* A \psi_n d\tau$$

instead of

$$A_{mn} = \int \psi_m^* A \psi_n d\tau$$

taken over the entire configuration space, where ψ_m, ψ_n are eigenfunctions belonging to the m-th and n-th states respectively

In the solution of problems of Quantum Mechanics such integrals occur which in the last analysis reduces to

$$\int^{\infty} k_0^{2n} \cos k_0 a \, dk_0, \quad \int^{\infty} k_0^{2n+1} \sin k_0 a \, dk_0$$

$$\int^{\infty} k_0^{2n+1} \cos k_0 a \, dk_0. \int^{\infty} k_0^{2n} \sin k_0 a \, dk_0$$

[*cf.* Dirac, (1943)]

In the customary process these integrals have to be calculated in an artificial manner. But our theory gives the values of these integrals in a natural and logical manner. We get, for $\alpha \geq$, $2n+1$, $2n+2$, $2n+2$, $2n+1$ respectively,

$$(C, \alpha)\text{-}\int_0^{\infty} k_0^{2n} \cos k_0 a dk_0 = 0, \quad (C, \alpha)\text{-}\int_0^{\infty} k_0^{2n+1} \sin k_0 a \, dk_0 = 0$$

$$(C, \alpha)\text{-}\int_0^{\infty} k_0^{2n+1} \cos k_0 a \, dk_0 = (-1)^{n+1}/k_0^{2n+2}, \quad (C, \alpha)\text{-}\int_0^{\infty} k_0^{2n} \sin k_0 a \, dk_0 = (-1)^n.(2n)!/k_0^{2n+1}$$

Although we proved the continuity, differentiability and integrability properties in all generality *i.e.* $\alpha (> 0)$ can be an integer, rational or irrational quantity, in this paper we have given certain results for only positive integral values of α. The general technique for such cases will be published later.

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ON CERTAIN INTEGRALS INVOLVING BESSEL FUNCTIONS

By

B. N. Bose

(*Received February 20, 1948*)

The present note may be taken as a continuation of my previous note (Bose, 1945). The method of Operational Calculus has been employed here as well. It is worthwhile to note that the changes in the order of integration wherever effected can be easily justified on account of the absolute convergence of the double integrals concerned.

1. It can be easily deduced from the relation

$$\int_0^\infty J_\mu(az)J_\nu(az)z^{\lambda+1}\exp(-z^2t^2).dz = \frac{a^{\mu+\nu}}{2^{\mu+\nu}t^{\lambda+\mu+\nu}}.\frac{\Gamma\{\frac{1}{2}(\lambda+\mu+\nu)\}}{\Gamma(\mu+1)\Gamma(\nu+1)}$$

$$\times {}_3F_3\left[\begin{matrix}\frac{1}{2}(\mu+\nu+1), & \frac{1}{2}(\lambda+\mu+\nu+2), & \frac{1}{2}(\lambda+\mu+\nu); & -a^2/t^2 \\ \mu+1, & \nu+1, & \mu+\nu+1; & \end{matrix}\right],$$

where $\mathcal{R}(\lambda+\mu+\nu) > 0$, that

$$\int_0^\infty J_m^2(az)z\exp(-z^2t).dz = (1/t)\exp(-a^2/2t).I_m(a^2/2t), \text{ provided } \mathcal{R}(m) > -1. \quad (1)$$

In Hankel's integral (1875) (Watson, 1922, p. 429)

$$\int_0^\infty \frac{zJ_m^2(yz)dz}{z^2+k^2} = I_m(ky)K_m(ky),$$

where $\mathcal{R}(k) > 0$, $y > 0$ and m, any number. On putting $k = \sqrt{p}$ and multiplying both sides by p and interpreting the left hand side by (1), we get

$$pI_m(y\sqrt{p}).K_m(y\sqrt{p}) \doteqdot (1/t)\exp(-y^2/2t).I_m(y^2/2t), \quad \mathcal{R}(m) > -1. \quad (2)$$

It has been proved (Bose, 1944) for $m > -\frac{1}{2}$, that

$$\int_0^1 P_n(1-2y^2)I_m(ky)K_m(ky)y^{2m+1}dy$$

$$= \frac{(-1)^n\Gamma(2m+1)}{2}\sum_{r=0}^{n}\frac{(-n)_r(n+1)_r\Gamma(2m+1+r)}{\Gamma(r+1)\Gamma(2m-n+1+r)\Gamma(2m+n+2+r)}$$

$$\times[I_{m+r}(k)K_{m+r}(k)+I_{m+r+1}(k)K_{m+r+1}(k)]. \quad (3)$$

We put $k = \sqrt{p}$ in (3) and multiply both sides by p. On interpreting both sides then and writing y for y^2 and a for $1/(2t)$, we get

$$\int_0^1 P_n(1-2y^2)\exp(-ay^2).I_m(ay^2)y^{2m+1}dy$$

$$= \frac{(-1)^n\Gamma(2m+1)}{2}\sum_{r=0}^{n}\frac{(-n)_r(n+1)_r\Gamma(2m+1+r)}{\Gamma(r+1)\Gamma(2m-n+1+r)\Gamma(2m+n+2+r)}.e^{-a}$$

$$\times[I_{m+r}(a)+I_{m+r+1}(a)]. \quad m \geq -\tfrac{1}{2}; \quad (4)$$

when $m = 0$, we get

$$\int_0^1 P_n(1-2y^2) \exp(-ay^2).I_0(ay^2)y\,dy = \frac{1}{2(2n+1)}.\exp(-a)[I_n(a)+I_{n+1}(a)], \tag{5}$$

which has already been obtained by a different method (Bose, 1945).

Again, we have

$$K_m(z) = \frac{\Gamma(\frac{1}{2})}{2^m\Gamma(m+\frac{1}{2})} \int_0^\infty z^m \sinh^{2m}\varphi \exp(-z\cosh\varphi).d\varphi, \text{ for } \mathcal{R}(m) > -\tfrac{1}{2}$$

On multiplying both sides by $z^{m'}$ and writing $z = \sqrt{p}$, we get

$$(\sqrt{p})^{m'}K_m(\sqrt{p}) = \frac{\Gamma(\frac{1}{2})}{2^m\Gamma(m+\frac{1}{2})} \int_0^\infty (\sqrt{p})^{m+m'} \sinh^{2m}\varphi \exp(-\sqrt{p}.\cosh\varphi).d\varphi. \tag{6}$$

We have due to Goldstein (1931)

$$(\sqrt{p})^{\frac{1}{2}\mu} \exp(-\sqrt{p}.r) \doteqdot (2/\pi)^{\frac{1}{2}}(1/2t)^{\frac{1}{2}\mu}.D_{\mu-1}\{r/\sqrt{(2t)}\}.\exp(-r^2/4t)$$

and

$$(\sqrt{p})^{\mu}.K_\nu(\sqrt{p}) \doteqdot (1/t)^{\frac{1}{2}(\mu-1)}W_{\frac{1}{2}(\mu-1),\frac{1}{2}\nu}(1/4t).$$

Interpreting both sides and making use of Goldstein's results and writing a for $1/\sqrt{(2t)}$, we get

$$\int_0^\infty \exp(-\tfrac{1}{4}a^2\cosh^2\varphi).D_{m+m'-1}(a\cosh\varphi)\sinh^{2m}\varphi\,d\varphi$$

$$= \frac{\Gamma(m+\frac{1}{2})2^{2m+m'-2}}{a^{m+1}}.\exp(-\tfrac{1}{4}a^2).W_{\frac{1}{2}(m'-1),\frac{1}{2}m}(\tfrac{1}{2}a^2), \quad (a>0) \tag{7}$$

Writing $m' = 2s+2$ and $m = 0$ in (7), we get, when s is a positive integer

$$\int_0^\infty \exp(-\tfrac{1}{4}a^2\cosh^2\varphi).D_{2s+1}(a\cosh\varphi)d\varphi = (-1)^s\sqrt{\pi}.2^{s-\frac{1}{2}}\exp(-\tfrac{1}{4}a^2).L_s(\tfrac{1}{2}a^2) \tag{8}$$

On writing $m' = 2s+m+2$, where s and m are positive integers, we get from (7)

$$\int_0^\infty \exp(-\tfrac{1}{4}a^2\cosh^2\varphi).D_{2s+2m+1}(a\cosh\varphi).\sinh^{2m}\varphi\,d\varphi$$

$$= 2^{m+s-1/2}\Gamma(m+\tfrac{1}{2})\Gamma(s+1)\Gamma(m+s+1)\exp(-\tfrac{1}{4}a^2).T_m^s(\tfrac{1}{2}a^2),$$

where $T_m^s(z)$ is defined to be Sonine's polynomial (Whittaker and Watson, 1927, p. 352).

The integral on the L. H. S. of (7) can be expressed in terms of the Pearson-Cunningham function $\omega_{n,m}(z)$ (Whittaker and Watson, 1927, p. 353).

We have (Watson, 1922)

$$\int_0^\infty J_{2\nu}(yt)\exp(-a^2t^2)dt = \frac{\sqrt{\pi}}{2a}\exp(-y^2/8a^2).I_\nu(y^2/8a^2), \quad \mathcal{R}(\nu) > -\tfrac{1}{2}. \tag{9}$$

We put $\nu = 0$ and multiply both sides by $P_n(1-2y^2)y$ and integrate with respect to y

between zero and one. Changing the order of integration on the left hand side, we get by (5)

$$\int_0^\infty J_{2n+1}(t) \exp(-a^2t^2).\frac{dt}{t} = \frac{\sqrt{\pi}}{4(2n+1)}\cdot(1/a)\exp(-1/8a^2).[I_n(1/8a^2)+I_{n+1}(1/8a^2)] \tag{10}$$

We have from Sonine's first finite integral (Watson, 1922)

$$J_{2n+1}(z)/z = \int_0^{\pi/2} J_{2n}(z\sin\theta)\sin^{2n+1}\theta\cos\theta d\theta \tag{11}$$

Let us multiply both sides of (11) by $\exp(-a^2z^2)$ and integrate with respect to z between zero and infinity. Changing the order of integration on the right hand side and then using (9), we get, on writing a for $1/8a^2$ and $\sin\theta$ for $\sin^2\theta$,

$$\int_0^{\pi/2}\exp(-a\sin\theta)I_n(a\sin\theta)\sin^n\theta\cos\theta d\theta = \frac{1}{2n+1}\cdot\exp(-a)[I_n(a)+I_{n+1}(a)] \tag{12}$$

We have again (Bose, 1944)

$$\int_0^\infty J_{2n+2}[y\sqrt{(t^2+z^2)}]\frac{tdt}{t^2+z^2} = \frac{1}{(n+1)}\frac{1}{yz}\sum_{r=0}^{n}(2r+1)J_{2r+1}(yz) \tag{13}$$

We put $z=1$ and multiply both sides by $\exp(-a^2y^2)$ and integrate with respect to y between zero and infinity. Changing the order of integration on the left hand side and using (9) and (10), we get, on writing $t = \tan\theta$,

$$\int_0^{\pi/2}\exp(-a\tan^2\theta).I_{n+1}(a\sec^2\theta)\tan\theta d\theta = \frac{1}{2(n+1)}\cdot\sum_{r=0}^{n}[I_r(a)+I_{r+1}(a)],\ (a>0) \tag{14}$$

Treating similarly the relation

$$\int_0^\infty J_{2n+2}[y\sqrt{(t^2+z^2)}]\frac{tdt}{(t^2+z^2)^{n+1}} = \frac{J_{2n+1}(yz)}{yz^{2n+1}}$$

which is obtained from Sonine's integral (Watson, 1922), we get

$$\int_0^{\pi/2}\exp(-a\tan^2\theta).I_{n+1}(a\sec^2\theta)\cos^{2n-1}\theta\sin\theta d\theta = \frac{1}{2(2n+1)}[I_n(a)+I_{n+1}(a)] \tag{15}$$

Now, put $\nu = \frac{1}{2}$ in (9) and multiply both sides by $P_n(1-2y^2)y^2$ and integrate with respect to y between the limits zero and one. Changing the order of integration on the left hand side and making use of (4) and the relation (Bose, 1944)

$$\int_0^1 P_n(1-2y^2)J_1(yt)y^2dy = \frac{1}{2n+1}\cdot\frac{1}{t}[(n+1)J_{2n+2}(t)-nJ_{2n}(t)],$$

we get

$$(n+1)\int_0^\infty \exp(-a^2t^2).J_{2n+2}(t)\frac{dt}{t} - n\int_0^\infty \exp(-a^2t^2).J_{2n}(t)\frac{dt}{t}$$
$$= (\sqrt{\pi}/8a).\exp(-1/8a^2).[I_{n+3/2}(1/8a^2)-I_{n-1/2}(1/8a^2)]. \tag{16}$$

Putting $n-1$, $n-2$, . . . , 0 for n in succession in (16) and adding the results, we get

$$\int_0^\infty \exp\,(-a^2t^2).J_{2n+2}(t)\,\frac{dt}{t} = \frac{\sqrt{\pi}}{4(2n+2)}\cdot\frac{\exp.(-1/8a^2)}{a}\sum_{r=0}^{n}\,[I_{r+3/2}(1/8a^2)-I_{r-1/2}(1/8a^2)]. \quad (17)$$

Treating the relation

$$\frac{1}{z}\,J_{2n+2}(z) = \int_0^{\pi/2} J_{2n+1}(z\,\sin\,\theta)\,\sin^{2n+2}\theta\,\cos\,\theta d\theta$$

which is obtained from Sonine's first finite integral, in exactly the same way as (11) and on making use of (17), we get, after writing a for $1/(8a^2)$, and $\sin\,\theta$ for $\sin^2\theta$,

$$\int_0^{\pi/2} \exp(-a\sin\theta).I_{n+1/2}(a\sin\,\theta)\,\sin^{n+1/2}\theta\,\cos\,\theta d\theta = \frac{1}{2n+2}\cdot\exp(-a)\sum_{r=0}^{n}[I_{r+3/2}(a)-I_{r-1/2}(a)] \quad (18)$$

Again treating the relation

$$\int_0^\infty J_{2n+3}[y\sqrt{(t^2+1)}]\,\frac{tdt}{(1+t^2)^{n+3/2}} = \frac{1}{y}\,J_{2n+2}(y)$$

as before and on making use of (17) we get, as in (15)

$$\int_0^{\pi/2} \exp\,(-a\,\tan^2\theta).I_{n+3/2}(a\,\sec^2\theta)\,\cos^{2n}\theta\,\sin\,\theta d\theta$$

$$= \frac{1}{2(2n+2)}\sum_{r=0}^{n}\,[I_{r+3/2}(a)-I_{r-1/2}(a)], \quad (a > 0). \qquad (19)$$

2. We now give a list of Bessel functions each of which satisfies a homogeneous integral equation of which the nucleus is $\{\sin\,a(z+t)\}/\{\pi(z+t)\}$. The importance of a function satisfying such an equation lies in its use in certain inversion formula found out by Hardy (1909) and Bateman (1906).

We start with Gallop's Integral (1886)

$$\int_{-\infty}^\infty \frac{\sin\,a(z+t)}{\pi(z+t)}\,J_0(bt)dt = J_0(bz), \quad (a \geqq b > 0) \qquad (20)$$

It has already been proved (Bose, 1944) that

$$\int_{-\infty}^\infty \frac{\sin\,a(z+t)}{\pi(z+t)}.\frac{J_{2n+1}(bt)}{t}\,dt = \frac{J_{2n+1}(bz)}{z}, \quad (a \geqq b > 0)\,. \qquad (21)$$

In (20) let us write $b\,\sin\,\theta$ for b and multiply both sides by $\cos^{2m-1}\theta\,\sin\,\theta$ and integrate with respect to θ between zero and $\pi/2$. Changing the order of integration which can be easily justified in this and in similar subsequent cases and using the relation

$$\int_0^{\pi/2} J_0(z\,\sin\,\theta)\,\cos^{2m-1}\theta\,\sin\,\theta d\theta = \frac{2^{m-1}\Gamma(m)}{z^m}\,J_m(z), \quad (m > 0)$$

which is obtained from Sonine's first finite integral, we get

$$\int_{-\infty}^{\infty} \frac{\sin a(z+t)}{\pi(z+t)} \frac{J_m(bt)}{t^m} dt = \frac{J_m(bz)}{z^m}, \quad (a \geqq b > 0,\ m \geqq -\tfrac{1}{2}). \tag{22}$$

By putting $m = -\frac{1}{2}$ and $m = \frac{1}{2}$, we get

$$\int_{-\infty}^{\infty} \frac{\sin a(z+t)}{\pi(z+t)} \cos bt\, dt = \cos bz, \quad (a \geqq b > 0) \tag{23}$$

$$\int_{-\infty}^{\infty} \frac{\sin a(z+t)}{\pi(z+t)} \frac{\sin bt}{t} dt = \frac{\sin bz}{z}, \quad (a \geqq b > 0) \tag{24}$$

Let us write $2bx$ for b in (23) and (24) and then multiply both sides of each by $P_n(1-2x^2)$ and integrate with respect to x between zero and one. Changing the order of integration and making use of the relations (Bose, 1944)

$$\int_0^1 P_n(1-2x^2) \cos (2xz) dx = \frac{\pi}{2} (-1)^n J_{n+1/2}(z) J_{-n-1/2}(z)$$

and

$$\int_0^1 P_n(1-2x^2) \sin (2xz) dx = \frac{\pi}{2} J^2_{n+1/2}(z),$$

we get

$$\int_{-\infty}^{\infty} \frac{\sin a(z+t)}{\pi(z+t)} J_{n+1/2}(bt) J_{-n-1/2}(bt) dt = J_{n+1/2}(bz) J_{-n-1/2}(bz), \quad (a \geqq 2b > 0) \tag{25}$$

$$\int_{-\infty}^{\infty} \frac{\sin a(z+t)}{\pi(z+t)} J^2_{n+1/2}(bt) dt = J^2_{n+1/2}(bz), \quad (a \geqq 2b > 0) \tag{26}$$

Let us multiply both sides of (21) by $\sin^{2n+2}\theta \cos^{2\nu+1}\theta$, write $b \sin\theta$ for b and integrate with respect to θ between zero and $\pi/2$. Using Sonine's first finite integral (Watson, 1922, p. 373)

$$\frac{z^{\nu+1}}{2^\nu \Gamma(\nu+1)} \int_0^{\pi/2} J_\mu(z \sin \theta) \sin^{\mu+1} \theta \cos^{2\nu+1}\theta d\theta = J_{\mu+\nu+1}(z), \ \mathcal{R}(\mu) > -1,\ \mathcal{R}(\nu) > -1,$$

we get, after writing ν for $\nu+2$

$$\int_{-\infty}^{\infty} \frac{\sin a(z+t)}{\pi(z+t)} \frac{J_{2n+\nu}(bt)}{t^\nu} dt = \frac{J_{2n+\nu}(bz)}{z^\nu}, \quad (a \geqq b > 0,\ \nu > -\tfrac{1}{2}). \tag{27}$$

It can be easily deduced from (27) (by writing $2b \cos\theta$ for b, integrating with respect to θ between zero and $\pi/2$ and applying Neumann's result after changing the order of integration).

$$\int_{-\infty}^{\infty} \frac{\sin a(z+t)}{\pi(z+t)} \frac{J^2_{n+\nu/2}(bt)}{t^\nu} dt = \frac{J^2_{n+\nu/2}(bz)}{z^\nu}, \quad (a \geqq 2b > 0,\ \nu > -\tfrac{1}{2}) \tag{28}$$

Again in (22) by writing $2m$ for m and $2b \cos\theta$ for b, we deduce as in (28),

$$\int_{-\infty}^{\infty} \frac{\sin a(z+t)}{\pi(z+t)} \frac{J_m^2(bt)}{t^{2m}}\, dt = \frac{J_m^2(bz)}{z^{2m}}, \quad (a \geqq 2b > 0,\ m > -\tfrac{1}{2}) \tag{29}$$

It is obvious that (29) is a particular case of (28). In (24) let us write $b\cos\theta$ for b and multiply both sides by $\sin^{2m}\theta$. Integrating with respect to θ between zero and $\pi/2$, we get

$$\int_{-\infty}^{\infty} \frac{\sin a(z+t)}{\pi(z+t)} \frac{H_m(bt)}{t^{m+1}}\, dt = \frac{H_m(bz)}{z^{m+1}}, \quad (a \geqq b \geq 0,\ m \geq -\tfrac{1}{2}). \tag{30}$$

As a special case we get

$$\int_{-\infty}^{\infty} \frac{\sin a(z+t)}{\pi(z+t)} \frac{1-\cos bt}{t^2}\, dt = \frac{1-\cos bz}{z^2} \tag{31}$$

Let us write $b\sin\theta$ for b in (22) and multiply both sides by $J_\nu(bx\cos\theta)\sin^{m+1}\theta\cos^{\nu+1}\theta$ and integrate with respect to θ between the limits zero and $\pi/2$. On making use of Sonine's second finite integral (Watson, 1922, p. 376), we get

$$\int_{-\infty}^{\infty} \frac{\sin a(z+t)}{\pi(z+t)} \frac{J_{m+\nu+1}[b\sqrt{(x^2+t^2)}]dt}{(x^2+t^2)^{(m+\nu+1)/2}}$$
$$= \frac{J_{m+\nu+1}[b\sqrt{(x^2+z^2)}]}{(x^2+z^2)^{(m+\nu+1)/2}}, \quad (a \geqq b > 0,\ m \geq -\tfrac{1}{2},\ \nu > -1) \tag{32}$$

Putting $m+\nu+1=0$ and writing by for b, we multiply both sides by $yP_n(1-2y^2)$. Integrating with respect to y between the limits zero and one, we get

$$\int_{-\infty}^{\infty} \frac{\sin a(z+t)}{\pi(z+t)} \frac{J_{2n+1}[b\sqrt{(x^2+t^2)}]dt}{\sqrt{(x^2+t^2)}} = \frac{J_{2n+1}[b\sqrt{(x^2+z^2)}]}{\sqrt{(x^2+z^2)}}, \quad (a \geqq b \geq 0) \tag{33}$$

Let us again multiply both sides of (22) by $\cos\{\tfrac{1}{2}w(1-b^2)\}b^{m+1}$ and integrate with respect to b between the limits zero and one. Changing the order of integration and making use of the integral representation of Lommel's function (Watson, 1922, p. 540) *viz.*,

$$U_\nu(w, z) = \frac{w^\nu}{z^{\nu-1}} \int_0^1 J_{\nu-1}(zt)\cos\{\tfrac{1}{2}w(1-t^2)\}t^\nu dt, \quad \mathcal{R}(\nu) > 0,$$

we get, on writing m for $m+1$

$$\int_{-\infty}^{\infty} \frac{\sin a(z+t)}{\pi(z+t)} U_m(w, t)dt = U_m(w, z), \quad (m > \tfrac{1}{2},\ a \geqq 1).$$

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ON THE DIVISORS OF $a\alpha^n + b\beta^m$.

By

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(Communicated by S. S. Pillai, Received March 12, 1948)

In his paper "On the divisors of a^n+1", Pillai has established the theorem that no prime of the form $4am-1$ divides a^n+1 for any n. The object of this note is to obtain a similar and more general result which can be stated as follows—

If a, b, α, β be four positive coprime integers and $q = ab\alpha\beta/Q^2$, where Q^2 is the greatest square dividing $ab\alpha\beta$, then no integer of the form $4kq-1$ divides $a\alpha^n + b\beta^m$ for any pair of non-negative integral values of n and m.

Let $\xi = a\alpha^n + b\beta^m$. Then clearly ξ is an integer which can be represented primitively by the binary quadratic form ax^2+by^2, $a\alpha x^2+by^2$, $ax^2+b\beta y^2$ or $a\alpha x^2+b\beta y^2$ according as n and m are both even; n odd, m even; n even, m odd; or both n, m are odd. The discriminant D_i of these four forms are $D_1 = -ab$, $D_2 = -ab\alpha$, $D_3 = -ab\beta$, $D_4 = -ab\alpha\beta$. Put $q_i = -D_i/Q_i^2$, where Q_i^2 is the greatest square dividing D_i. We know that if ξ is an integer representable primitively by a binary quadratic form with discriminant D and $p \mid \xi$, then $\left(\frac{D}{p}\right) = 1$ where $\left(\frac{D}{p}\right)$ is the Legendre's symbol. On the other hand, if $\left(\frac{D}{P}\right) = -1$ where $\left(\frac{D}{P}\right)$ is Jacobi's symbol then clearly not $P \mid \xi$.

Suppose now

$$P_i = 4kq_i - 1 \quad i = 1, 2, 3, 4.$$

Then

$$\left(\frac{D_i}{P_i}\right) = \left(\frac{-q_iQ_i^2}{P_i}\right) = -\left(\frac{q_i}{P_i}\right)$$

(*i*) If $q_i = 4\lambda+1$, then

$$\left(\frac{q_i}{P_i}\right) = \left(\frac{P_i}{q_i}\right) = \left(\frac{-1}{q_i}\right) = 1.$$

(*ii*) If $q_i = 4\lambda-1$, then

$$\left(\frac{q_i}{P_i}\right) = -\left(\frac{P_i}{q_i}\right) = -\left(\frac{-1}{q_i}\right) = 1$$

(*iii*) If $q_i = 2q_i'$ where q_i' is odd, then

$$\left(\frac{q_i}{P_i}\right) = \left(\frac{2}{P_i}\right)\left(\frac{q_i'}{P_i}\right) = \left(\frac{2}{P_i}\right) = 1$$

Hence

$$\left(\frac{D}{P_i}\right) = -1.$$

This implies that no integer of the form $4kq_i-1$ divides ξ when ξ can be represented by the *i* th quadratic form given above. Plainly the set of numbers $4kq_4-1$ is a subset of each of the sets $4kq_1-1$, $4kq_2-1$, $4kq_3-1$.

Hence we get the result stated above.

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ON THE BENDING OF AN ELLIPTIC PLATE UNDER CERTAIN DISTRIBUTIONS OF LOAD—(I)*

By

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(*Communicated by Prof. N. M. Basu—Received March 1, 1948*)

In the year 1921, Happel determined the deflexion of the central plane of a thin elliptic plate made of isotropic elastic, material, loaded at the centre and clamped or supported at the edge. This was followed by a paper by S. Ghosh (1924) where the latter obtained the deflexion with the load placed at a focus, the edge being supposed clamped. In the present paper the author proposes to determine the deflexion of the central plane of a thin elliptic plate under the action of a load concentrated at any point on the straight line joining the foci. The edge will be supposed to be clamped.

It may be remarked that the results obtained by Ghosh and also those due to Happel for the clamped edge case are thus particular cases of the results contained in this paper

Let the xy-plane coincide with the mid-plane of the elliptic plate in its unstretched state. We take the origin at the centre of the bounding ellipse and suppose that the axes of x and y are coincident with its major and minor axes respectively. The positive direction of the z-axis is taken upwards. *i.e.*, opposite to the direction towards which the weight acts.

Let the cartesian equation of the bounding ellipse be given by

$$x^2/a^2+y^2/b^2 = 1, \quad (a > b > 0).$$

We introduce elliptic co-ordinates by the transformation

$$x+iy = c\cosh(\xi+i\eta), \quad (c > 0) \tag{1}$$

It follows that

$$x = c\cosh\xi\cos\eta$$

$$y = c\sinh\xi\sin\eta$$

and

$$\frac{1}{h^2} = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 = \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 = \frac{c^2}{2}(\cosh 2\xi - \cos 2\eta).$$

The curves $\xi =$ const. and $\eta =$ const. represent a system of confocal ellipses and hyperbolas respectively. The centre of any ellipse $\xi =$ const. is given by $\xi = 0$, $\eta = \pi/2$ and the foci are given by $\xi = 0$, $\eta = 0$; and $\xi = 0$, $\eta = \pi$.

In particular if $\xi = \alpha$ be the equation of the bounding ellipse, then $\alpha \geq 0$ and $a = c\cosh\alpha$, $b = c\sinh\alpha$ and $c^2 = a^2 - b^2$.

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* The author has also worked out the case when the load is placed anywhere on the upper face of the plate.

We shall suppose the elliptic plate to be bounded by the curve $\xi = \alpha$ and the load to be placed at the point $\xi = 0$, $\eta = \beta$ where $0 \leqq \beta \leqq \pi$. The cartesian co-ordinates of the point where the load is placed are therefore given by $(c \cos \beta, 0)$.

The problem will stand solved if we can determine a continuous function w:

(1) which satisfies the differential equation

$$\nabla_1^4 w = 0$$

where

$$\nabla_1^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \tag{2}$$

throughout the plate excepting the point $(0, \beta)$ where the load is supposed to have been placed;

(2) which satisfies the boundary conditions $w = 0$ and $\partial w/\partial \nu = 0$ over $\xi = \alpha$, ν being the direction of the inward-drawn normal at any point of the bounding ellipse;

(3) which yields finite and continuous stress resultant and stress couples throughout the plate excepting at the point $(0, \beta)$; and

(4) which is such that if the normal shearing stress across a curve s be denoted by*

$$N = -D(\partial/\partial\nu)\nabla_1^2 w$$

then we must have

$$\int_C N ds = W \tag{3}$$

where C is any closed curve lying entirely within the ellipse with the point $(0, \beta)$ in its interior.

We may, however, note that across the ellipse $\xi = \alpha$

$$\partial w/\partial \nu = -h(\partial w/\partial \xi)$$

where

$$1/h^2 = \tfrac{1}{2}c^2(\cosh 2\xi - \cos 2\eta)$$

Now over $\xi = \alpha$, $(\alpha > 0)$, $\cosh 2\xi > 1$. So over $\xi = \alpha$, $(\alpha > 0)$, $h^2 > 0$. So the condition $\partial w/\partial \nu = 0$ over $\xi = \alpha$ reduces to $\partial w/\partial \xi = 0$ over $\xi = \alpha$. Thus the boundary conditions at a clamped edge, *viz.*, $w = 0$, $\partial w/\partial \nu = 0$ over $\xi = \alpha$ reduce to $w = 0$ and $\partial w/\partial \xi = 0$ over $\xi = \alpha$.

Now let (x, y) denote the cartesian co-ordinates of any point of the mid plane and let us put

$$r^2 = (x - c \cos \beta)^2 + y^2.$$

Further, let us assume for the deflexion

$$w = -\frac{W}{8\pi D} r^2(\log r - \log \tfrac{1}{2}c) + w'(x, y) \tag{4}$$

---

* $D = \frac{2}{3}Ek^3/(1-\sigma^2)$, where $2k$ is the thickness of the plate and E and σ are the Young's modulus and Poisson's ratio of the elastic material.

where $w'(x, y)$ is some continuous function satisfying the differential equation

$$\nabla_1^4 w' = 0$$

throughout the plate. Now

$$\nabla_1^4[r^2(\log r - \log \tfrac{1}{2}c)] = 0$$

at every point of the plate except the point $(0, \beta)$.

Denoting the contribution to the normal shearing stress resultant due to the function

$$-\frac{W}{8\pi D}r^2(\log r - \log \tfrac{1}{2}c)$$

by N_1, we have across a circular arc $r =$ const.

$$N_1 = W/2\pi r.$$

So

$$\int_C N_1 ds = W$$

where C is any circle of radius r with the point $(0, \beta)$ as centre lying entirely within the ellipse. The above is also true, even if, C be any closed curve with the point $(0, \beta)$ in its interior.

If N_2 denotes the contribution to the normal shearing stress due to the function $w'(x, y)$, i.e., if

$$N_2 = -D(\partial/\partial\nu)\nabla_1^2 w'$$

then

$$\int_C N_2 ds = -D\int_C \frac{\partial}{\partial\nu}\nabla_1^2 w' ds = -D\iint_\Gamma \nabla_1^4 w' dx\,dy = 0.$$

In the above, C stands for any closed curve lying entirely within the ellipse and Γ denotes the region contained within the closed curve C. It follows that

$$\int_{C_1} N\,ds = \int_{C_1}(N_1+N_2)ds = W$$

and

$$\int_{C_2} N\,ds = \int_{C_2}(N_1+N_2)ds = 0$$

where C_1 is any closed curve containing the point $(0, \beta)$ in its interior, while C_2 is any closed curve with $(0, \beta)$ lying wholly exterior to it.

Thus the function

$$w = -\frac{W}{8\pi D}r^2(\log r - \log \tfrac{1}{2}c) + w'(x, y)$$

with properly adjusted $w'(x, y)$ so as to satisfy the boundary conditions would be the solution of the problem in hand.

We shall now adjust $w'(x, y)$ in such a manner that

$$w = \partial w/\partial\xi = 0 \quad \text{over} \quad \xi = \alpha.$$

We proceed as follows:

$$\begin{aligned} r^2 &= (x-c\cos\beta)^2+y^2 \\ &= c^2[(\cosh\xi\cos\eta-\cos\beta)^2+\sinh^2\xi\sin^2\eta] \\ &= \tfrac{1}{4}c^2\exp 2\xi.[\{1-\exp\{-i(\eta-\beta)\}.\exp(-\xi)\}\{1-\exp i(\eta-\beta).\exp(-\xi)\} \\ &\qquad\times\{1-\exp\{-i(\eta+\beta)\}.\exp(-\xi)\}\{1-\exp i(\eta+\beta).\exp(-\xi)\}] \end{aligned}$$

(E. Mathieu, 1869). So,

$$\begin{aligned} \log r^2 = \log(\tfrac{1}{2}ce^{\xi})^2 &+ \log[1-\exp\{-i(\eta-\beta)\}.\exp(-\xi)]+\log\{1-\exp i(\eta-\beta).\exp(-\xi)\} \\ &+\log[1-\exp\{-i(\eta+\beta)\}.\exp(-\xi)]+\log\{1-\exp i(\eta+\beta)\exp(-\xi)\}. \end{aligned}$$

We can easily show that

$$\log r = \log\tfrac{1}{2}c+\xi-2\sum_{n=1}^{\infty}\frac{\exp(-n\xi).\cos\eta\beta\cos n\eta}{n} \tag{5}$$

The series converges absolutely for all η and β and for all $\xi > 0$. It converges uniformly for all η and β and for all $\xi \geqq \epsilon$ where $\epsilon > 0$ is arbitrary.

We have therefore, for $\xi > 0$,

$$\begin{aligned} r^2(\log r-\log\tfrac{1}{2}c) = \tfrac{1}{4}c^2\Big[&\{\xi(2\cosh 2\xi+2+2\cos 2\beta)+2(1+\exp(-2\xi))+(2+\exp(-2\xi)).\cos 2\beta\} \\ &-\{8\xi\cosh\xi\cos\beta+2\cos\beta\exp\xi+(7\cos\beta+\cos 3\beta)\exp(-\xi) \\ &\qquad+(\cos\beta-\tfrac{1}{3}\cos 3\beta)\exp(-3\xi)\}\cos\eta \\ &+\{2\xi+(2+\cos 2\beta)+(1+\tfrac{2}{3}\cos 2\beta-\tfrac{1}{3}\cos 4\beta)\exp(-2\xi) \\ &\qquad+(-\tfrac{1}{3}\cos 2\beta+\tfrac{1}{6}\cos 4\beta)\exp(-4\xi)\}\cos 2\eta \\ &+2\sum_{n=3}^{\infty}\Big\{\Big(-\frac{\cos(n-2)\beta}{n-2}+\frac{2\cos\beta\cos(n-1)\beta}{n-1}-\frac{\cos n\beta}{n}\Big)\exp\{-(n-2)\xi\} \\ &+\Big(\frac{2\cos\beta\cos(n-1)\beta}{n-1}-\frac{2(1+\cos 2\beta)\cos n\beta}{n}+\frac{2\cos\beta\cos(n+1)\beta}{n+1}\Big)\exp(-n\xi) \\ &+\Big(-\frac{\cos n\beta}{n}+\frac{2\cos\beta\cos(n+1)\beta}{n+1}-\frac{\cos(n+2)\beta}{n+2}\Big)\exp\{-(n+2)\xi\}\Big\}\cos n\eta\Big] \end{aligned} \tag{6}$$

We assume for the deflexion the expression

$$\begin{aligned} w = -\frac{W}{8\pi D}[r^2(\log r-\log\tfrac{1}{2}c)]+A'_0+A_0\cosh 2\xi+(A'_1\cosh\xi+A_1\cosh 3\xi)\cos\eta \\ +\sum_{n=2}^{\infty}\{A_{n-2}\cosh(n-2)\xi+A'_n\cosh n\xi+A_n\cosh(n+2)\xi\}\cos n\eta^{*} \end{aligned} \tag{7}$$

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* The expression $E=A_0\cosh 2\xi+\sum_{n=2}^{\infty}\{A_{n-2}\cosh(n-2)\xi+A_n\cosh(n+2)\xi\}\cos n\eta$ is a formal solution of the differential equation $\nabla_1^4E=0$ in elliptic co-ordinates. It was given in essentially the above form by H. Happel (1921). See also Timpe (1923).

The constants A's and A''s have now to be determined so that the boundary conditions may be true on $\xi = \alpha$.

For $\xi \geq 0$ the above expression may be written as

$$w = -\frac{Wc^2}{32\pi D}\Big[\xi(2\cosh 2\xi+2+2\cos 2\beta)+2(1+\cos 2\beta)+(2+\cos 2\beta)\exp(-2\xi)$$

$$-\{8\xi\cosh\xi\cos\beta+2\cos\beta\exp\xi+(7\cos\beta+\cos 3\beta)\exp(-\xi)$$

$$+(\cos\beta-\tfrac{1}{3}\cos 3\beta)\exp(-3\xi)\}\cos\eta$$

$$+\{2\xi+(2+\cos 2\beta)+(1+\tfrac{2}{3}\cos 2\beta-\tfrac{1}{3}\cos 4\beta)\exp(-2\xi)$$

$$+(-\tfrac{1}{3}\cos 2\beta+\tfrac{1}{6}\cos 4\beta)\exp(-4\xi)\}\cos 2\eta$$

$$+2\sum_{n=3}^{\infty}\Big\{\Big(-\frac{\cos(n-2)\beta}{n-2}+\frac{2\cos\beta\cos(n-1)\beta}{n-1}-\frac{\cos n\beta}{n}\Big)\exp\{-(n-2)\xi\}$$

$$+\Big(\frac{2\cos\beta\cos(n-1)\beta}{n-1}+\frac{2(1+\cos 2\beta)\cos n\beta}{n}+\frac{2\cos\beta\cos(n+1)\beta}{n-1}\Big)\exp(-n\xi)$$

$$+\Big(-\frac{\cos n\beta}{n}+\frac{2\cos\beta\cos(n+1)\beta}{n+1}-\frac{\cos(n+2)\beta}{n+2}\Big)\exp\{-(n+2)\xi\}\Big\}\cos n\eta\Big]$$

$$+A'_0+A_0\cosh 2\xi+(A'_1\cosh\xi+A_1\cosh 3\xi)$$

$$+\sum_{n=2}^{\infty}\{A_{n-2}\cosh(n-2)\xi+A'_n\cosh n\xi+A_n\cosh(n+2)\xi\}\cos n\eta$$

By formal differentiation with respect to ξ term by term we have

$$\frac{\partial w}{\partial\xi} = -\frac{Wc^2}{32\pi D}\Big[2\{(\cosh 2\xi+1+\cos 2\beta)+2\xi\sinh 2\xi-2\exp(-2\xi)-\exp(-2\xi).\cos 2\beta\}$$

$$-\{8(\cosh\xi+\xi\sinh\xi)\cos\beta+2\exp\xi.\cos\beta-\exp(-\xi).(7\cos\beta+\cos 3\beta)$$

$$-\exp(-3\xi).(3\cos\beta-\cos 3\beta)\}\cos\eta$$

$$+2\{1-\exp(-2\xi).(1+\tfrac{2}{3}\cos 2\beta-\tfrac{1}{3}\cos 4\beta)-2\exp(-4\xi).(-\tfrac{1}{3}\cos 2\beta+\tfrac{1}{6}\cos 4\beta)\}\cos 2\eta$$

$$-2\sum_{n=3}^{\infty}\Big\{(n-2)\exp\{-(n-2)\xi\}.\Big(-\frac{\cos(n-2)\beta}{n-2}+\frac{2\cos\beta\cos(n-1)\beta}{n-1}-\frac{\cos n\beta}{n}\Big)$$

$$+n\exp(-n\xi).\Big(\frac{2\cos\beta\cos(n-1)\beta}{n-1}-\frac{2(1+\cos 2\beta)\cos n\beta}{n}+\frac{2\cos\beta\cos(n+1)\beta}{n+1}\Big)$$

$$+(n+2)\exp\{-(n+2)\xi\}.\Big(-\frac{\cos n\beta}{n}+\frac{2\cos\beta\cos(n+1)\beta}{n+1}-\frac{\cos(n+2)\beta}{n+2}\Big)\Big\}\cos n\eta\Big]$$

$$+2A_0\sinh 2\xi+(A'_1\sinh\xi+3A_1\sinh 3\xi)\cos\eta$$

$$+\sum_{n=2}^{\infty}\{(n-2)A_{n-2}\sinh(n-2)\xi+nA'_n\sinh n\xi+(n+2)A_n\sinh(n+2)\xi\}\cos n\eta.$$

We shall later on prove that the sum of the above series is the correct expression for $\partial w/\partial\xi$ for $0 < \xi < 2\alpha$.

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The boundary conditions $w = 0$ and $\partial w/\partial\xi = 0$ over $\xi = \alpha$ give rise to two sequences of linear equations involving the constants A'_0, A_0, A'_1, A_1, etc. These equations are easily solved.

The boundary conditions $w = \partial w/\partial\nu = 0$ over $\xi = \alpha$, thus enable us to determine the values of the constants A's. In fact we have

$$A'_0 \sinh 2\alpha = \frac{Wc^2}{32\pi D}[(2+\cos 2\beta)+2(1+\alpha)(1+\cos 2\beta)\sinh 2\alpha-(1+\cos 2\beta)\cosh 2\alpha-\cosh^2 2\alpha] \tag{9}$$

$$A_0 \sinh 2\alpha = \frac{Wc^2}{32\pi D}[(1+\cos 2\beta)-\exp(-2\alpha).(2+\cos 2\beta)+\cosh 2\alpha+2\alpha \sinh 2\alpha] \tag{10}$$

$$\begin{aligned}A'_1(2 \sinh 2\alpha+\sinh 4\alpha) = -\frac{Wc^2}{32\pi D}&[3(\cos\beta-\tfrac{1}{3}\cos 3\beta)+2\exp\alpha.(3\sinh 2\alpha-\cosh 3\alpha)\cos\beta\\&+\exp(-\alpha).(7\cos\beta+\cos 3\beta)(3\sinh 3\alpha+\cosh 3\alpha)+8(3\alpha\sinh 3\alpha\cos\beta\\&-\cosh 3\alpha\cos\beta)\cosh\alpha-8\alpha\sinh\alpha\cosh 3\alpha\cos\beta]\end{aligned} \tag{11}$$

$$\begin{aligned}A_1(2\sinh 2\alpha+\sinh 4\alpha) = \frac{Wc^2}{32\pi D}&[(5\cos\beta+\cos 3\beta)-8\cosh^2\alpha\cos\beta\\&+\exp(-3\alpha).(3\cosh\alpha+\sinh\alpha)(\cos\beta-\tfrac{1}{3}\cos 3\beta)]\end{aligned} \tag{12}$$

$$\begin{aligned}A'_2(3\sinh 2\alpha+\sinh 6\alpha) = -4A_0\sinh 4\alpha+\frac{Wc^2}{16\pi D}&[\{4\alpha+2(2+\cos 2\beta)\\&+2\exp(-2\alpha).(1+\tfrac{2}{3}\cos 2\beta-\tfrac{1}{3}\cos 4\beta)-2\exp(-4\alpha).(\tfrac{1}{3}\cos 2\beta-\tfrac{1}{6}\cos 4\beta)\}\sinh 4\alpha\\&-\{1-\exp(-2\alpha).(1+\tfrac{2}{3}\cos 2\beta-\tfrac{1}{3}\cos 4\beta)+2\exp(-4\alpha).(\tfrac{1}{3}\cos 2\beta-\tfrac{1}{6}\cos 4\beta)\cosh 4\alpha\}]\end{aligned} \tag{13}$$

$$\begin{aligned}A_2(3\sinh 2\alpha+\sinh 6\alpha) = 2A_0\sinh 2\alpha-\frac{Wc^2}{16\pi D}&[\{2\alpha+(2+\cos 2\beta)\\&+\exp(-2\alpha).(1+\tfrac{2}{3}\cos 2\beta-\tfrac{1}{3}\cos 4\beta)-\exp(-4\alpha).(\tfrac{1}{3}\cos 2\beta-\tfrac{1}{6}\cos 4\beta)\}\sinh 2\alpha\\&-\{1-\exp(-2\alpha).(1+\tfrac{2}{3}\cos 2\beta-\tfrac{1}{3}\cos 4\beta)+2\exp(-4\alpha).(\tfrac{1}{3}\cos 2\beta-\tfrac{1}{6}\cos 4\beta)\}\cosh 2\alpha]\end{aligned} \tag{14}$$

$$\begin{aligned}A'_n[(n+1)\sinh 2\alpha+\sinh(2n+2)\alpha] = &-A_{n-2}[n\sinh 4\alpha+2\sinh 2n\alpha]\\&+\frac{Wc^2}{16\pi D}[\exp\{-(n-2)\alpha\}.B_n^{(1)}\{(n+2)\sinh(n+2)\alpha+(n-2)\cosh(n+2)\alpha\}\\&+\exp(-n\alpha).B_n^{(2)}\{(n+2)\sinh(n+2)\alpha+n\cosh(n+2)\alpha\}+(n+2)B_n^{(3)}]\end{aligned} \tag{15}$$

$$\begin{aligned}A_n[(n+1)\sinh 2\alpha+\sinh(2n+2)\alpha] = &A_{n-2}[(n-1)\sinh 2\alpha+\sinh(2n-2)\alpha]\\&-\frac{Wc^2}{16\pi D}[\exp\{-(n-2)\alpha\}.B_n^{(1)}\{n\sinh n\alpha+(n-2)\cosh n\alpha\}+nB_n^{(2)}\\&+\exp\{-(n+2)\alpha\}.B_n^{(3)}\{n\sinh n\alpha+(n+2)\cosh n\alpha\}]\end{aligned} \tag{16}$$

where for $n \geqq 3$

$$B_n^{(1)} = -\frac{\cos(n-2)\beta}{n-2}+\frac{2\cos\beta\cos(n-1)\beta}{n-1}-\frac{\cos n\beta}{n} \tag{17}$$

$$B_n^{(2)} = \frac{2\cos\beta\cos(n-1)\beta}{n-1} - \frac{2(1+\cos 2\beta)\cos n\beta}{n} + \frac{2\cos\beta\cos(n+1)\beta}{n+1} \tag{18}$$

$$B_n^{(3)} = -\frac{\cos n\beta}{n} + \frac{2\cos\beta\cos(n+1)\beta}{n+1} - \frac{\cos(n+2)\beta}{n+2} \tag{19}$$

It is easy to verify that we obtain Ghosh's result* from the above by putting $\beta = 0$. Also if we put $\beta = \pi/2$ in our result we get the same expression for w as given by Happel only with the sign reversed. This difference in sign arises from the fact that the positive directions of the z-axis have been taken in the opposite directions in the two papers.

We have now to consider the convergence of the series that appears in the expression for the deflexion and examine the properties of the function defined by it. We therefore take up the series

$$w' = A'_0 + A_0\cosh 2\xi + (A'_1\cosh\xi + A_1\cosh 3\xi)\cos\eta$$
$$+ \sum_{n=2}^{\infty}\{A_{n-2}\cosh(n-2)\xi + A'_n\cosh n\xi + A_n\cosh(n+2)\xi\}\cos n\eta \tag{20}$$

appearing in the formula (7) purported to give the deflexion, where the values of the constants A'_n and A_n are given by the formulae (9), (10), . . ., (19). We shall prove that w' represents a function that is finite and continuous throughout the plate including the boundary. We shall further show that the contributions to the stress resultant N and stress couples G and H due to it are finite and continuous throughout the plate including the boundary and that it satisfies the equation

$$\nabla_1^4 w' = 0 \tag{21}$$

throughout the plate.

It is also necessary to justify the process which was employed to get the values of the constants A'_n, A_n, etc.

In order to achieve our object it is first necessary to determine the orders of magnitudes of the co-efficients A'_n and A_n. In the following section we shall therefore determine two inequalities giving the orders of A'_n and A_n.

We first consider the constants $B_n^{(1)}$, $B_n^{(2)}$ and $B_n^{(3)}$ given by the formulæ (17), (18) and (19) respectively. We have for $n \geqq 3$

$$B_n^{(1)} = -\frac{\cos(n-2)\beta}{n-2} + \frac{2\cos\beta\cos(n-1)\beta}{n-1} - \frac{\cos n\beta}{n} = \frac{\cos(n-2)\beta}{(n-1)(n-2)} + \frac{\cos n\beta}{n(n-1)} \tag{22}$$

Therefore

$$|B_n^{(1)}| \leqq 2/\{n(n-2)\} \tag{23}$$

$$B_n^{(2)} = \frac{2\cos\beta\cos(n-1)\beta}{-1} - \frac{2(1+\cos 2\beta)\cos n\beta}{n} + \frac{2\cos\beta\cos(n+1)\beta}{n+1}$$

$$= \frac{\cos(n-2)\beta}{n(n-1)} + \frac{2\cos n\beta}{(n-1)n(n+1)} - \frac{\cos(n+2)\beta}{n(n+1)} \tag{24}$$

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* It seems that there are a few misprints in Ghosh's paper. It is only at those places that the two results do not agree.

Therefore

$$|B_n^{(2)}| \leqq 2/\{n(n-2)\} \tag{25}$$

Again

$$B_n^{(3)} = -\frac{\cos n\beta}{n}+\frac{2\cos\beta\cos(n+1)\beta}{n+1}-\frac{\cos(n+2)\beta}{n+2} = -\frac{\cos n\beta}{n(n+1)}+\frac{\cos(n+2)\beta}{(n+1)(n+2)} \tag{26}$$

Therefore

$$|B_n^{(3)}| \leqq 2/\{n(n+2)\} \tag{27}$$

Now, writing

$$f_n = (n+1)\sinh 2\alpha+\sinh(2n+2)\alpha$$

we have

$$A_nf_n = A_{n-2}f_{n-2}-\frac{Wc^2}{16\pi D}\Big[\exp(2\alpha).B_n^{(1)}\Big\{\frac{n}{2}(1-\exp(-2n\alpha))+\frac{n-2}{2}(1+\exp(-2n\alpha))\Big\}$$
$$+nB_n^{(2)}+\exp(-2\alpha).B_n^{(3)}\Big\{\frac{n}{2}(1-\exp(-2n\alpha))+\frac{n+2}{2}(1+\exp(-2n\alpha))\Big\}\Big]$$

$$= A_{n-2}f_{n-2}-\frac{Wc^2}{16\pi D}\,[\exp(2\alpha).B_n^{(1)}\{(n-1)-\exp(-2n\alpha)\}+nB_n^{(2)}$$
$$+\exp(-2\alpha).B_n^{(3)}\{(n+1)+\exp(-2n\alpha)\}] \tag{28}$$

We may, therefore, write the following

$$A_{2n}f_{2n} = A_{2n-2}f_{2n-2}-\frac{Wc^2}{16\pi D}\,[\exp(2\alpha)B_{2n}^{(1)}\{(2n-1)-\exp(-4n\alpha)\}+2nB_{2n}^{(2)}$$
$$+\exp(-2\alpha).B_{2n}^{(3)}\{(2n+1)+\exp(-4n\alpha)\}] \tag{29}$$

$$A_{2n-2}f_{2n-2} = A_{2n-4}f_{2n-4}-\frac{Wc^2}{16\pi D}\,[\exp(2\alpha)B_{2n-2}^{(1)}\{(2n-3)-\exp(-(4n-4)\alpha)$$
$$+(2n-2)B_{2n-2}^{(2)}+\exp(-2\alpha).B_{2n-2}^{(3)}\{(2n-1)+\exp(-(4n-4)\alpha\}] \tag{30}$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

$$A_8f_8 = A_6f_6-\frac{Wc^2}{16\pi D}\,[\exp(2\alpha)B_8^{(1)}\{7-\exp(-16\alpha)\}+8B_8^{(2)}+\exp(-2\alpha).B_8^{(3)}\{9+\exp(-16\alpha)\}] \tag{31}$$

$$A_6f_6 = A_4f_4-\frac{Wc^2}{16\pi D}\,[\exp(2\alpha)B_6^{(1)}\{5-\exp(-12\alpha)\}+6B_6^{(2)}+\exp(-2\alpha).B_6^{(3)}\{7+\exp(-12\alpha)\}] \tag{32}$$

$$A_4f_4 = A_2f_2-\frac{Wc^2}{16\pi D}\,[\exp(2\alpha)B_4^{(1)}\{3-\exp(-8\alpha)\}+4B_4^{(2)}+\exp(-2\alpha).B_4^{(3)}\{5+\exp(-8\alpha)\}] \tag{33}$$

Adding, we have

$$A_{2n}f_{2n} = A_2f_2-\frac{Wc^2}{16\pi D}\,[\exp(2\alpha).\{3B_4^{(1)}+5B_6^{(1)}+7B_8^{(1)}+\cdots+(2n-1)B_{2n}^{(1)}\}$$
$$-\{B_4^{(1)}\exp(-6\alpha)+B_6^{(1)}\exp(-10\alpha)+B_8^{(1)}\exp(-12\alpha)+\cdots+B_{2n}^{(1)}\exp(-(4n-2)\alpha)\}$$
$$+\{4B_4^{(2)}+6B_6^{(2)}+8B_8^{(2)}+\cdots+2nB_{2n}^{(2)}\}$$
$$+\exp(-2\alpha).\{5B_4^{(3)}+7B_6^{(3)}+9B_8^{(3)}+\cdots+(2n+1)B_{2n}^{(3)}\}$$
$$+\{B_4^{(3)}\exp(-10\alpha)+B_6^{(3)}\exp(-14\alpha)+B_8^{(3)}\exp(-18\alpha)+\cdots+B_{2n}^{(3)}\exp(-(4n+2)\alpha)\}] \tag{34}$$

We may show that

$$3B_4^{(1)} = -\frac{\cos 2\beta}{2} + \frac{\cos 4\beta}{4}$$

$$5B_6^{(1)} = -\frac{\cos 4\beta}{4} + \frac{\cos 6\beta}{6}$$

$$7B_8^{(1)} = -\frac{\cos 6\beta}{6} + \frac{\cos 8\beta}{8}$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$(2n-1)B_{2n}^{(1)} = -\frac{\cos(2n-2)\beta}{2n-2} + \frac{\cos 2n\beta}{2n}.$$

Hence

$$3B_4^{(1)} + 5B_6^{(1)} + 7B_8^{(1)} + \cdots + (2n-1)B_{2n}^{(1)} = -\frac{\cos 2\beta}{2} + \frac{\cos 2n\beta}{2n}$$

which gives

$$|3B_4^{(1)} + 5B_6^{(1)} + 7B_8^{(1)} + \cdots + (2n-1)B_{2n}^{(1)}| \leqq \tfrac{1}{2} + (1/2n) < 1 \quad \text{for} \quad n \geqq 2 \tag{35}$$

Also, since $\alpha > 0$, we have $0 < \exp(-\alpha) < 1$. Therefore

$$|B_4^{(1)}\exp(-6\alpha) + B_6^{(1)}\exp(-10\alpha) + B_8^{(1)}\exp(-14\alpha) + \cdots + B_{2n}^{(1)}\exp\{-(4n-2)\alpha\}|$$
$$\leqq |B_4^{(1)}| + |B_6^{(1)}| + |B_8^{(1)}| + \cdots + |B_{2n}^{(1)}| < \tfrac{1}{2} \tag{36}$$

for $n \geqq 2$.

Again, it is easy to show that

$$4B_4^{(2)} = \frac{\cos 2\beta}{3} + \frac{\cos 4\beta}{3} - \frac{\cos 4\beta}{5} - \frac{\cos 6\beta}{5}$$

$$6B_6^{(2)} = \frac{\cos 4\beta}{5} + \frac{\cos 6\beta}{5} - \frac{\cos 6\beta}{7} - \frac{\cos 8\beta}{7}$$

$$8B_8^{(2)} = \frac{\cos 6\beta}{7} + \frac{\cos 8\beta}{7} - \frac{\cos 8\beta}{9} - \frac{\cos 10\beta}{9}$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

$$2nB_{2n}^{(2)} = \frac{\cos(2n-2)\beta}{2n-1} + \frac{\cos 2n\beta}{2n-1} - \frac{\cos 2n\beta}{2n+1} - \frac{\cos(2n+2)\beta}{2n+1}$$

Adding the above, we have

$$4B_4^{(2)} + 6B_6^{(2)} + 8B_8^{(2)} + \cdots + 2nB_{2n}^{(2)} = \frac{\cos 2\beta}{3} + \frac{\cos 4\beta}{3} - \frac{\cos 2n\beta}{2n+1} - \frac{\cos(2n+2)\beta}{2n+1}$$

Therefore

$$|4B_4^{(2)} + 6B_6^{(2)} + 8B_8^{(2)} + \cdots + 2nB_{2n}^{(2)}| \leqq \tfrac{1}{3} + \tfrac{1}{3} + 1/(2n+1) + 1/(2n+1) < 1 \text{ for } n \geqq 3 \tag{37}$$

It is easy to shew that

$$5B_4^{(3)} + 7B_6^{(3)} + 9B_8^{(3)} + \cdots + (2n+1)B_{2n}^{(3)} = -\frac{\cos 4\beta}{4} + \frac{\cos(2n+2)\beta}{2n+2}.$$

So

$$|5B_4^{(3)}+7B_6^{(3)}+9B_8^{(3)}+\cdots+(2n+1)B_{2n}^{(3)}|<\tfrac{1}{2}, \quad \text{for} \quad n \geqq 2.$$

Again

$$|B_4^{(3)}\exp(-6\alpha)+B_6^{(3)}\exp(-8\alpha)+B_8^{(3)}\exp(-10\alpha)+\cdots+B_{2n}^{(3)}\exp\{-(4n-2)\alpha\}|$$

$$\leqq \frac{2}{4\cdot 6}+\frac{2}{6\cdot 8}+\frac{2}{8\cdot 10}+\cdots+\frac{2}{2n(2n+2)}=\frac{1}{4}-\frac{1}{2n+2}<\frac{1}{4} \quad \text{for} \quad n \geqq 2 \qquad (38)$$

Collecting all these together, we have

$$|A_{2n}f_{2n}| \leqq |A_2 f_2| + \frac{Wc^2}{16\pi D}\{\exp(2\alpha)+\tfrac{7}{4}+\tfrac{1}{2}\exp(-2\alpha)\} \quad \text{for} \quad n \geqq 3.$$

It follows that there is a positive number P' independent of n such that

$$|A_{2n}f_{2n}|<P' \text{ for all } n.$$

In like manner, we can show that for all n

$$|A_{2n+1}f_{2n+1}|<P'' ,$$

where P'' is another fixed positive number.

If we write $\frac{1}{4}P$ for the greater of the two numbers P' and P'' we have

$$|A_n f_n|<\tfrac{1}{4}P \text{ for all } n.$$

Now

$$f_n=(n+1)\sinh 2\alpha+\sinh(2n+2)\alpha.$$

Since $\alpha>0$, we have $\sinh 2\alpha>0$ and $\sinh(2n+2)\alpha>0$ and so $f_n>\sinh(2n+2)\alpha$ $>\frac{1}{4}\exp(2n+2)\alpha$ for sufficiently large n. Therefore

$$|A_n|<P/\exp(2n+2)\alpha$$

for sufficiently large n. On the other hand we can easily show that for sufficiently large n,

$$|A'_n|<Q/\exp(2n+2)\alpha$$

where Q is a fixed positive number independent of n.

Now, for the consideration of its convergence, and the examination of the properties of the function defined by it, we break up the series

$$w'=A'_0+A_0\cosh 2\xi+(A'_1\cosh\xi+A_1\cosh 3\xi)\cos\eta$$
$$+\sum_{n=2}^{\infty}\{A_{n-2}\cosh(n-2)\xi+A'_n\cosh n\xi+A_n\cosh(n+2)\xi\}\cos n\eta$$

into two parts, *viz.*,

$$F_1=\sum_{n=0}^{\infty}A'_n\cosh n\xi\cos n\eta$$

and

$$F_2=A_0\cosh 2\xi+A_1\cosh 3\xi\cos\eta+\sum_{n=2}^{\infty}\{A_{n-2}\cosh(n-2)\xi+A_n\cosh(n+2)\xi\}\cos n\eta.$$

We further split up the series for F_2 into two parts, *viz.*,

$$F_{2,1} = \sum_{n=2}^{\infty} A_{n-2} \cosh (n-2)\xi \cos n\eta$$

and

$$F_{2,2} = \sum_{n=0}^{\infty} A_n \cosh (n+2)\xi \cos n\eta.$$

We first take the series

$$F_1 = \sum_{n=0}^{\infty} A'_n \cosh n\xi \cos n\eta$$

We have

$$|A'_n \cosh n\xi \cos n\eta| \leqq |A'_n| \cosh n\xi.$$

Now, for sufficiently large n

$$|A'_n| < Q/\exp(2n+2)\alpha,$$

where Q is a positive constant independent of n and

$$0 < \cosh n\xi \leqq \exp(n|\xi|)$$

So for sufficiently large n

$$|A'_n \cosh n\xi \cos n\eta| < Q \exp(n|\xi|)/\exp(2n+2)\alpha = Q \exp(-2\alpha).\exp n(|\xi|-2\alpha)$$

The series

$$F_1 = \sum_{n=0}^{\infty} A'_n \cosh n\xi \cos n\eta$$

therefore converges absolutely in ξ, η for all η and for all ξ such that $|\xi| < 2\alpha$. It converges uniformly in ξ and η for all η and for all ξ such that $|\xi| \leqq 2\alpha-\epsilon$. where $\epsilon(\epsilon > 0)$ is arbitrary. At any rate the series is absolutely and uniformly convergent in both the variables within the given elliptic plate including the boundary $\xi = \alpha$. The sum F_1 therefore represents a function that is continuous throughout the plate including the boundary.

We next consider the series

$$F_{2,1} = \sum_{n=2}^{\infty} A_{n-2} \cosh (n-2)\xi \cos n\eta$$

Now,

$$|A_{n-2} \cosh(n-2)\xi \cos n\eta| \leqq |A_{n-2}| \cosh(n-2)\xi$$

$$< \frac{P \cosh (n-2)\xi}{\exp(2n-2)\alpha} \leqq \frac{P \exp (n-2)|\xi|}{\exp(2n-2)\alpha} = P \exp(-2\alpha).\exp(n-2)(|\xi|-2\alpha)$$

Here P is a positive constant independent of n.

The series for $F_{2,1}$ is therefore absolutely convergent in ξ and η for all η and all ξ such that $|\xi| < 2\alpha$. It converges uniformly in ξ and η for all η and for all ξ such that $|\xi| \leqq 2\alpha-\epsilon$ where $\epsilon(\epsilon > 0)$ is small at pleasure. In particular the series converges absolutely and uniformly in both variables throughout the whole plate including the boundary $\xi = \alpha$. The series therefore represents a continuous function throughout the plate.

Similarly we can show that the series

$$F_{2,2} = \sum_{n=0}^{\infty} A_n \cosh(n+2)\xi \cos n\eta$$

behaves in exactly the same way as the previous two series in the matter of its nature and region of convergence. Its sum therefore represents a continuous function throughout the plate including the boundary.

The auxiliary series

$$w' = A'_0 + A_0 \cosh 2\xi + (A'_1 \cosh \xi + A_1 \cosh 3\xi)\cos \eta$$

$$+ \sum_{n=2}^{\infty} \{A_{n-2} \cosh(n-2)\xi + A'_n \cosh n\xi + A_n \cosh(n+2)\xi\} \cos n\eta .$$

which is the sum of the three series F_1 and $F_{2,1}$ and $F_{2,2}$ is thus absolutely convergent in ξ and η for all η and for all ξ such that $|\xi| < 2\alpha$. It is uniformly convergent in ξ and η for all η and for all ξ such that $|\xi| \leqq 2\alpha - \epsilon$ where $\epsilon (> 0)$ is small at pleasure. The series is therefore absolutely and uniformly convergent in both the variables throughout the entire plate including the boundary $\xi = \alpha$ The series may therefore be re-arranged in any way we like and its sum which remains unaltered for any re-arrangement of the terms represents a continuous function throughout the plate.

We now consider the series that is obtained by term by term differentiation of the series for w' with respect to ξ The process gives rise to the following new series, *viz.*,

$$f = 2A_0 \sinh 2\xi + (A'_1 \sinh \xi + 3A_1 \sinh 3\xi)\cos \eta + (2A'_2 \sinh 2\xi + 4A_2 \sinh 4\xi)\cos 2\eta$$

$$+ \sum_{n=3}^{\infty} \{(n-2)A_{n-2} \sinh(n-2)\xi + nA'_n \sinh n\xi + (n+2)A_n \sinh(n+2)\xi\} \cos n\eta$$

As before, we have

$$|nA'_n \sinh n\xi \cos n\eta| < \frac{nQ \exp n|\xi|}{\exp(2n+2)\alpha} = nQ \exp(-2\alpha).\exp n(|\xi| - 2\alpha)$$

where n is sufficiently large and Q is a positive constant independent of n.

The series $\sum nA'_n \sinh n\xi \cos n\eta$ therefore converges absolutely in ξ and η for all η and for all ξ such that $|\xi| < 2\alpha$.

It converges uniformly in ξ and η for all η and for all ξ such that $|\xi| \leqq 2\alpha - \epsilon$, where $\epsilon (> 0)$ is small at pleasure. It follows that the series $\sum nA'_n \sinh n\xi \cos n\eta$ converges absolutely and uniformly throughout the elliptic plate including its boundary. We can further prove that each of the series

$$\sum_{n=2}^{\infty} (n-2)A_{n-2} \sinh(n-2)\xi \cos n\eta$$

and

$$\sum_{n=0}^{\infty} (n+2)A_n \sinh(n+2)\xi \cos n\eta$$

converges absolutely and uniformly in ξ and η throughout the elliptic plate including the boundary. The series f is thus uniformly convergent throughout the elliptic plate including the boundary $\xi = \alpha$. So its sum represents the correct value of $\partial w'/\partial \xi$ throughout the elliptic plate including the boundary. In fact it represents $\partial w'/\partial\xi$ in the interior of the bigger ellipse $0 \leqq \xi < 2\alpha$, and is continuous throughout that range. The same argument may be employed to prove that partial derivatives of all orders with respect to ξ and η of w' may be had by term-by-term differentiation and they represent continuous functions throughout the plate including the boundary.

The series for w' being proved to be convergent in the interior of the ellipse $\xi = 2\alpha$, we may put $\xi = \alpha$ in each term of the series and the sum is the value of $w'(\alpha)$. The value of $(\partial w'/\partial\xi)_{\xi=\alpha}$ is equal to the sum of the series obtained by differentiating the series for w' term-by-term with respect to ξ and then putting $\xi = \alpha$ in the derived series. This happens on account of the uniform convergence of the derived series throughout the interior as well as on the boundary of the ellipse $\xi = 2\alpha-\epsilon$, where $\epsilon(> 0)$ is small at pleasure.

The series for $r^2(\log r-\log \tfrac{1}{2}c)$ converges absolutely in ξ and η for all η and all ξ such that $\xi > 0$. It converges uniformly in ξ and η for all η and all $\xi \geqq \epsilon$ where $\epsilon(> 0)$ is small at pleasure. The value of $r^2(\log r-\log \tfrac{1}{2}c)$ is thus the sum of the series for it when we put $\xi = \alpha$ in each term of the series. On the other hand the series obtained by differentiating term-by-term the series for $r^2(\log r-\log \tfrac{1}{2}c)$ with respect to ξ converges absolutely in ξ and η for all η and all $\xi > 0$. The new series also converges uniformly in ξ and η for all η and and all $\xi \geqq \epsilon$, where $\epsilon(> 0)$ is small at pleasure. The sum of the derived series therefore represents the function $\partial\{r^2(\log r-\log \tfrac{1}{2}c)\}/\partial\xi$ for all $\xi > 0$. In particular, the value of the latter for $\xi = \alpha$ is obtained by putting $\xi = \alpha$ in each term of the derived series.

The above considerations demonstrate the validity of the process employed to get the values of the constants A'_n, A_n.

It follows that the expression for w as given by formula (7) represents a function continuous throughout the plate including the boundary, except at the point $\xi = 0$, $\eta = \beta$ where it is not defined. However, w tends to a definite finite limit as r tends to zero and ascribing to w this limiting value at $\xi = 0$, $\eta = \beta$, w happens to be continuous throughout the plate.

That w satisfies the boundary conditions $w = 0$ and $\partial w/\partial\xi = 0$ over $\xi = \alpha$ has just been shown.

Again, following the method adopted by Happel we can easily show that the function w' satisfies the differential equation

$$\nabla_1^4 w' = 0$$

throughout the plate excepting at the foci of the ellipse. The same method may be employed to show that the contributions to the stress resultant N and the stress-couples G and H due to function w' are continuous throughout the plate including the boundary excepting at the foci. The difficulty at the foci arises on account of the appearance of positive integral powers of the function h as factors in the various expressions of the stresses and also in the expression for $\nabla_1^4 w'$.

We therefore propose to follow an entirely different line of arguments to settle these points.* We shall use the concept of analytic functions of a complex variable z and shall show that this method succeeds in proving that w' satisfies the differential equation

$$\nabla_1^4 w' = 0$$

throughout the plate, the foci not excepted.

Also the method will establish the continuity of the contributions to the stresses due to the function w' throughout the entire plate including the boundary, the foci not excepted.

The plan referred to above is given in the following section.

Starting with the transformation

$$z = x+iy = c\cdot\cosh(\xi+i\eta) \quad (c>0).$$

We have on putting $\zeta = \xi+i\eta$

$$\cosh\xi\cos\eta+i\sinh\xi\sin\eta = \cosh\zeta = z/c,$$

$$\cosh 2\xi\cos 2\eta+i\sinh 2\xi\sin 2\eta = \cosh 2\zeta = \tfrac{1}{2}\{\exp(2\zeta)+\exp(-2\zeta)\} = (2z^2-c^2)/c^2,$$

$$\cosh 3\xi\cos 3\eta+i\sinh 3\xi\sin 3\eta = \cosh 3\zeta = (4z^3-3c^2z)/c^3$$

In like manner we can show that

$$\cosh 4\xi\cos 4\eta+i\sinh 4\xi\sin 4\eta = (8z^4-8c^2z^2+c^4)/c^4$$

We thus see that $\cosh\zeta$, $\cosh 2\zeta$, $\cosh 3\zeta$, $\cosh 4\zeta$ are polynomials in z of degrees 1, 2, 3 and 4 respectively.

We can now prove by induction that

$$\cosh n\zeta = \cosh n\xi\cos n\eta+i\sinh n\xi\sin n\eta$$

is a polynomial of degree n in z for all positive integral values of n.

To show this, we assume that it has been proved that $\cosh r\xi\cos r\eta+i\sinh r\xi\sin r\eta$ is a polynomial of degree r in z for $r = 1, 2, \ldots, n-1$. Then

$$\begin{aligned}\cosh n\xi\cos n\eta+i\sinh n\xi\sin n\eta &= \cosh n\zeta = \tfrac{1}{2}\{\exp n\zeta+\exp(-n\zeta)\}\\ &= \frac{1}{2}\Big[\{\exp\zeta+\exp(-\zeta)\}^n-\binom{n}{1}\{\exp(n-2)\zeta+\exp(-\overline{n-2}\zeta)\}\\ &\qquad -\binom{n}{2}\{\exp(n-4)\zeta+\exp(-\overline{n-4}\zeta)-\cdots-t_l\Big]\end{aligned}$$

where

$$t_l = \binom{n}{\frac{1}{2}n}$$

if n be even and

$$t_l = \binom{n}{\frac{1}{2}(n-1)}\{\exp\zeta+\exp(-\zeta)\}$$

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* The use of Cartesian co-ordinates was suggested by Prof. N M. Basu, and the author took up the idea of using the complex variable z from a study of the recent works of Mr. A. C. Stevenson (1942).

if n be odd. So,

$$\cosh n\xi \cos n\eta + i \sinh n\xi \sin n\eta = 2^{n-1}\cosh^n\zeta - \binom{n}{1}\cosh(n-4)\zeta - \binom{n}{2}\cosh(n-4)\zeta - \cdots - \tfrac{1}{2}t_l$$

where $\frac{1}{2}t_l$ is a constant if n be even and is equal to $\binom{n}{\frac{1}{2}(n-1)}\cosh\zeta$ if n be odd.

By hypothesis, $\cosh(n-2)\zeta$ is a polynomial of degree $(n-2)$ in z, $\cosh(n-4)\zeta$ is a polynomial of degree $(n-4)$ in z and so on.

Thus it is proved that $\cosh n\zeta$ is a polynomial of degree n in z. We proved that $\cosh n\zeta$ is a polynomial of degree n in z for $n = 1, 2, 3$ and 4. So, we can now assert that

$$\cosh n\zeta = \cosh n\xi \cos n\eta + i \sinh n\xi \sin n\eta$$

is a polynomial of degree n in z for all positive integral values of n.

On the other hand

$$\cosh 2\xi + \cos 2\eta = \frac{2z\bar{z}}{c^2} = \frac{2(x^2+y^2)}{c^2}$$

where we write $\bar{z}$ for $x - iy$. Again

$$(\cosh 4\xi \cos 2\eta + \cosh 2\xi \cos 4\eta) + i(\sinh 4\xi \sin 2\eta + \sinh 2\xi \sin 4\eta)$$

$$= 2(\cosh\xi \cos\eta - i \sinh\xi \sin\eta)(\cosh 3\xi \cos 3\eta + i \sinh 3\xi \sin 3\eta) = (2\bar{z}/c^4)(4z^3 - 3c^2z).$$

And

$$(\cosh 6\xi \cos 4\eta + \cosh 4\xi \cos 6\eta) + i(\sinh 6\xi \sin 4\eta + \sinh 4\xi \sin 6\eta)$$

$$= 2(\cosh\xi \cos\eta - i \sinh\xi \sin\eta)(\cosh 5\xi \cos 5\eta + i \sinh 5\xi \sin 5\eta) = \frac{2\bar{z}}{c} \cdot \frac{16z^5 - 20c^2z^3 + 5c^4z}{c^5}.$$

Also

$$\cosh 2n\xi \cos(2n-2)\eta + \cosh(2n-2)\xi \cos 2n\eta + i\{\sinh 2n\xi \sin(2n-2)\eta + \sinh(2n-2)\xi \sin 2n\eta\}$$

$$= 2(\cosh\xi \cos\eta - i \sinh\xi \sin\eta)(\cosh \overline{2n-1}\xi \cos \overline{2n-1}\eta + i \sinh \overline{2n-1}\xi \sin \overline{2n-1}\eta)$$

$$= \frac{2\bar{z}}{c^{2n}} P_{2n-1}(z)$$

where $P_{2n-1}(z)$ is a polynomial of degree $2n-1$ in z. Again,

$$\cosh 3\xi \cos\eta + \cosh\xi \cos 3\eta + i(\sinh 3\xi \sin 3\eta + \sinh\xi \sin 3\eta)$$

$$= 2(\cosh\xi \cos\eta - i \sinh\xi \sin\eta)(\cosh 2\xi \cos 2\eta + i \sinh 2\xi \sin 2\eta) = \frac{2\bar{z}}{c} \cdot \frac{2z^2 - c^2}{c^2}.$$

And

$$(\cosh 5\xi \cos 3\eta + \cosh 3\xi \cos 5\eta) + i(\sinh 5\xi \sin 3\eta + \sinh 3\xi \sin 5\eta)$$

$$= 2(\cosh\xi \cos\eta - i \sinh\xi \sin\eta)(\cosh 4\xi \cos 4\eta + i \sinh 4\xi \sin 4\eta) = \frac{2\bar{z}}{c} \cdot \frac{8z^4 - 8c^2z^2 + c^4}{c^4}.$$

Further,

$$\cosh(2n+1)\xi \cos(2n-1)\eta) + \cosh(2n-1)\xi \cos(2n+1)\eta$$
$$+ i\{\sinh(2n+1)\xi \sin(2n-1)\eta + \sinh(2n-1)\xi \sin(2n+1)\eta\}$$

$$= 2(\cosh\xi \cos\eta - i \sinh\xi \sin\eta)(\cosh 2n\xi \cos 2n\eta + i \sinh 2\xi \sin 2\eta) = \frac{2\bar{z}}{c} \cdot \frac{P_{2n}(z)}{c^{2n}}$$

where $P_{2n}(z)$ is a polynomial of degree $2n$ in z.

Now, the expression for the deflexion has been taken to be

$$w = -\frac{Wc^2}{32\pi D}r^2(\log r - \log \tfrac{1}{2}c) + A'_0 + A_0\cosh 2\xi + (A'_1\cosh\xi + A_1\cosh 3\xi)\cos\eta$$

$$+\sum_{n=2}^{\infty}\{A_{n-2}\cosh(n-2)\xi + A'_n\cosh n\xi + A_n\cosh(n+2)\xi\}\cos n\eta$$

We broke up the series into two parts, *viz* ,

$$F_1 = \sum_{n=0}^{\infty}A'_n\cosh n\xi\cos n\eta$$

and

$$F_2 = A_0\cosh 2\xi + A_1\cosh 3\xi\cos\eta + \sum_{n=2}^{\infty}\{A_{n-2}\cosh(n-2)\xi + A_n\cosh(n+2)\xi\}\cos n\eta.$$

Each of the series was proved to be absolutely convergent in ξ and η for all η and for all ξ such that $|\xi| < 2\alpha$. So also each was proved to be uniformly convergent for all η and for all ξ such that $|\xi| \leqq 2\alpha - \epsilon$, where $\epsilon(>0)$ is arbitrarily small.

We now write the series F_2 as

$$F_2 = \sum_{n=0}^{\infty}A_n\{\cosh(n+2)\xi\cos n\eta + \cosh n\xi\cos(n+2)\eta\}$$

which we can evidently do on account of the absolute convergence of the series throughout the interior of the ellipse $\xi = 2\alpha$. We have

$$F_1 = A'_0 + A'_1\cosh\xi\cos\eta + A'_2\cosh 2\xi\cos 2\eta$$
$$+ A'_3\cosh 3\xi\cos 3\eta + \cdots + A'_n\cosh n\xi\cos n\eta + \cdots$$

Let us write

$$F'_1 = A'_1\sinh\xi\sin\eta + A'_2\sinh 2\xi\sin 2\eta + A'_3\sinh 3\xi\sin 3\eta + \cdots + A'_n\sinh n\xi\sin n\eta + \cdots$$

It is easy to show that the series F'_1 converges absolutely in ξ and η for all η and for all ξ such that $|\xi| < 2\alpha$. It can also be shown to converge uniformly for all η and for all ξ such that $|\xi| \leqq 2\alpha - \epsilon$ where $\epsilon(>0)$ is small at pleasure.

Now, consider the complex series

$$F_1 + iF'_1 = A'_0 + A'_1(\cosh\xi\cos\eta + i\sinh\xi\sin\eta) + A'_2(\cosh 2\xi\cos 2\eta + i\sinh 2\xi\sin 2\eta)$$
$$+ A'_3(\cosh 3\xi\cos 3\eta + i\sinh 3\xi\sin 3\eta) + \cdots + A'_n(\cosh n\xi\cos n\eta + i\sinh n\xi\sin n\eta) + \cdots$$
$$= A'_0 + \frac{A'_1}{c}z + \frac{A'_2}{c^2}(2z^2 - c^2) + \frac{A'_3}{c^3}(4z^3 - 3c^2z) + \cdots + \frac{A'_n}{c^n}P_n(z) + \cdots$$

By what we have proved in regard to the convergence of F_1 and F'_1 separately it follows that the complex series is absolutely convergent for all η and for all ξ such that $|\xi| < 2\alpha$. Also it is uniformly convergent for all η and for all ξ such that $|\xi| \leqq 2\alpha - \epsilon$ where $\epsilon(>0)$ is arbitrarily small. It follows that the series in the complex variable z equivalent to $F_1 + iF'_1$ is certainly absolutely and uniformly convergent throughout the elliptic plate including the boundary. In point of fact it is absolutely and uniformly convergent in any closed domain lying entirely within the ellipse whose boundary is $\xi = 2\alpha$. On the other hand each term of the series, being a polynomial in z, is an analytic function throughout the entire z-plane. So the sum is an analytic function in the interior of any closed domain contained entirely within the ellipse $\xi = 2\alpha$.

Thus in any closed domain lying entirely within the ellipse $\xi = 2\alpha$, even if the domain happens to contain a focus in its interior, the real and imaginary parts of the above function are bounded and continuous possessing bounded and continuous partial derivatives of all orders with respect to x and y and also mixed derivatives. Thus contributions to the stress resultant N and stress-couples G and H due to the part F_1 of the deflexion-function are finite and continuous throughout the plate including the boundary. Also, each of the functions F_i and F'_1 is harmonic throughout the interior of the ellipse $\xi = 2\alpha$.

F_1 therefore satisfies the equation

$$\nabla_1^2 F_1 = 0$$

and so also $\nabla_1^4 F_1 = 0$, throughout the interior of the ellipse $\xi = 2\alpha$. Let us again write

$$F_2 = A_0 \cosh 2\xi + A_1 \cosh 3\xi \cos \eta + \sum_{n=2}^{\infty} \{A_{n-2} \cosh(n-2)\xi + A_n \cosh(n+2)\xi\} \cos n\eta$$

$$= A_0(\cosh 2\xi + \cos 2\eta) + A_1(\cosh 3\xi \cos \eta + \cosh \xi \cos 3\eta) + A_2(\cosh 4\xi \cos 2\eta$$
$$+ \cosh 2\xi \cos 4\eta) + \cdots + A_{2n}\{\cosh(2n+2)\xi \cos 2n\eta + \cosh 2n\xi \cos(2n+2)\eta\}$$
$$+ A_{2n+1}\{\cosh(2n+3)\xi \cos(2n+1)\eta + \cosh(2n+1)\xi \cos(2n+3)\eta\} + \cdots$$

This re-arrangement is permissible throughout the range of absolute convergence, so throughout the interior of the ellipse $\xi = 2\alpha$, hence certainly throughout the given elliptic plate including the boundary.

We now take the series

$$F'_2 = A_1(\sinh 3\xi \sin \eta + \sinh \xi \sin 3\eta) + A_2(\sinh 4\xi \sin 2\eta + \sinh 2\xi \sin 4\eta) + A_3(\sinh 5\xi \sin 3\eta$$
$$+ \sinh 3\xi \sin 5\eta) + \cdots + A_{2n}\{\sinh(2n+2)\xi \sin 2n\eta + \sinh 2n\xi \sin(2n+2)\eta\}$$
$$+ A_{2n+1}\{\sinh(2n+3)\xi \sin(2n+1)\eta + \sinh(2n+1)\xi \sin(2n+3)\eta\} + \cdots$$

It is easy to show that the series for F'_2 is absolutely convergent for all η and for all ξ such that $|\xi| < 2\alpha$. It is uniformly convergent for all η and for all ξ such that $|\xi| \leq 2\alpha - \epsilon$, where $\epsilon (> 0)$ is small at pleasure. The series is therefore absolutely convergent in ξ and η throughout any closed domain lying entirely within the ellipse $\xi = 2\alpha$.

We now consider the complex series

$$F_2 + iF'_2 = A_0(\cosh 2\xi + \cos 2\eta) + A_1\{(\cosh 3\xi \cos \eta + \cosh \xi \cos 3\eta) + i(\sinh 3\xi \sin \eta + \sinh \xi \sin 3\eta)\}$$
$$+ A_2\{(\cosh 4\xi \cos 2\eta + \cosh 2\xi \cos 4\eta) + i(\sinh 4\xi \sin 2\eta + \sinh 2\xi \sin 4\eta)\} + \cdots$$
$$+ A_{2n}\{(\cosh \overline{2n+2}\xi \cos 2n\eta + \cosh 2n\xi \cos \overline{2n+2}\eta) + i(\sinh \overline{2n+2}\xi \sin 2n\eta$$
$$+ \sinh n\xi \sin \overline{2n+2}\eta) + A_{2n+1}\{(\cosh \overline{2n+3}\xi \cos \overline{2n+1}\eta + \cosh \overline{2n+1}\xi \cos \overline{2n+3}\eta)$$
$$+ i(\sinh \overline{2n+3}\xi \sin \overline{2n+1}\eta + \sinh \overline{2n+1}\xi \sin 2\overline{n+3}\eta)\} + \cdots$$

By what has been proved already, we have

$$F_2+iF'_2 = \frac{2\bar{z}}{c}\left[\frac{A_0}{c}+\frac{A_1}{c^2}(2z^2-c^2)+\frac{A_2}{c^3}(4z^3-3c^2z)+\cdots+\frac{A_{2n}}{c^{2n+1}}P_{2n+1}(z)+\frac{A_{2n+1}}{c^{2n+2}}P_{2n+2}(z)+\cdots\right]$$

where $P_r(z)$ is some polynomial of degree r in z.

We have already remarked that the series for F_2 is absolutely convergent for all η and for all ξ such that $|\xi| < 2\alpha$. It is uniformly convergent for all η and all ξ such that $|\xi| \leqq 2\alpha-\epsilon$, where $\epsilon(>0)$ is small at pleasure. The same is also true of the series for F'_2. It follows that the complex series $F_2+iF'_2$ behaves exactly similarly in regard to its nature and domain of convergence

We shall show that the series within brackets is absolutely and uniformly convergent in any closed domain within the ellipse $\xi = 2\alpha$.

The series within brackets written in elliptic co-ordinates runs thus

$$A_0(\cosh\xi\cos\eta+i\sinh\xi\sin\eta)+A_1(\cosh 2\xi\cos 2\eta+i\sinh 2\xi\sin 2\eta)+A_2(\cosh 3\xi\cos 3\eta$$
$$+i\sinh 3\xi\sin 3\eta)+\cdots+A_n\{\cosh(n+1)\xi\cos(2n+1)\eta+i\sinh(n+1)\xi\sin(n+1)\eta\}+\cdots$$

We consider the series

$$\sum_{n=0}^{\infty} A_n\cosh(n+1)\xi\cos(n+1)\eta$$

and

$$\sum_{n=0}^{\infty} A_n\sinh(n+1)\xi\sin(n+1)\eta$$

It follows from arguments similar to those adopted previously that each of the series converges absolutely for all η and for all ξ such that $|\xi| < 2\alpha$. Also each series converges uniformly for all η and for all ξ such that $|\xi| \leqq 2\alpha-\epsilon$ where $\epsilon(>0)$ is arbitrarily small.

It follows that the series

$$\frac{A_0z}{c}+\frac{A_1}{c^2}(2z^2-c^2)+\frac{A_2}{c^3}(4z^3-3c^2z)+\cdots+\frac{A_n}{c^{n+1}}P_{n+1}(z)+\cdots$$

converges absolutely and uniformly throughout any closed domain lying entirely within the ellipse $\xi = 2\alpha$. The series therefore represents an analytic function of z, throughout the interior of the ellipse $\xi = 2\alpha$. The real and imaginary parts of the function defined by the series are thus continuous and possess continuous partial derivatives of all orders with respect to x and y throughout the interior of the ellipse $\xi = 2\alpha$. They also possess continuous mixed derivatives throughout the same range. Finally, the real and imaginary parts are each separately plane-harmonic throughout the interior of the ellipse $\xi = 2\alpha$. In symbols, if we write the sum of the series as $u+iu'$, we have

$$\nabla_1^2 u = \nabla_1^2 u' = 0$$

throughout the interior of the ellipse $\xi = 2\alpha$.

We may now write

$$F_2+iF'_2 = \frac{2(x-iy)}{c}(u+iu')$$

Therefore

$$F_2 = (2/c)(xu+yu').$$

So

$$\nabla_1{}^2F_2 = \frac{4}{c}\left(\frac{\partial u}{\partial x}+\frac{\partial u'}{\partial y}\right) = \frac{8}{c}\frac{\partial u}{\partial x}$$

$\left(\text{for } \frac{\partial u}{\partial x} = \frac{\partial u'}{\partial y}\right.$ throughout the interior of the eliipse $\left.\xi = 2\alpha\right)$ and therefore

$$\nabla_1{}^4F_2 = \frac{8}{c}\nabla_1{}^2\frac{\partial u}{\partial x} = \frac{8}{c}\frac{\partial}{\partial x}\nabla_1{}^2u = 0$$

throughout the interior of the ellipse $\xi = 2\alpha$.

The reversion of the order of differentiation is permissible on account of the continuity of all derivatives. F_2 is therefore biharmonic throughout the interior of the ellipse $\xi = 2\alpha$.

Thus the function $w'(x, y)$ which is represented by the sum of F_1+F_2 is continuous and biharmonic throughout the elliptic plate including the boundary. Besides it possesses finite and continuous partial derivatives of all orders with respect to x and y and also similar mixed derivatives throughout the elliptic plate $\xi = \alpha$, including the boundary. It follows that the contributions to the stress-resultant N and stress-couples G and H by this part of the deflexion function are finite and continuous throughout the elliptic plate including the boundary.

It follows that the formula (7) where the constants are given by the formulæ (9), (10), . . ., (19) represents the deflexion of the central plane of a thin elliptic plate clamped at the edge carrying a load of weight W concentrated at the point $\xi = 0$, $\eta = \beta$ $(0 \leq \beta \leq \pi)$ on the line joining the foci.

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BENDING OF RECTILINEAR PLATES

By

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Despite the wide attention that the important problem of the bending of plates has received not many complete solutions for rectilinear plates with supported edges have been obtained. The known cases are the rectangular plate, the equilateral plate and the right-angled isosceles triangular plate. In the last case the solution has been obtained in a double Fourier's series and then by an elaborate process reduced to a simple series (Nadai, 1925; Galerkin, 1919; Timoshenko, 1910). The equilateral plate has been dealt with by Woinowsky-Krieger (1933), Sen (1934) and Seth (1945).

We propose to develop a general method for supported rectilinear plates. The case of clamped plates has been discussed in a recent paper (Seth, 1947). It not only gives the known solutions for uniform pressure and uniformly varying pressure, but also suggests a general method when the pressure is a polynomial of the n-th degree. For uniform pressure it is shown that the solution can be made to depend on the corresponding torsion solution for the boundary.

Let w be the displacement of any point $(x, y, 0)$ of the middle plane of the plate of thickness $2h$, the face subjected to uniform pressure p being $z = -h$. Putting $D = \frac{2}{3}Eh^3/(1-\sigma^2)$, where E is the Young's modulus and σ the Poisson's ratio we know that the differential equation satisfied by w is (Love, 1927)

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 w = \nabla_1^4 w = (p/D) = k \text{ (say)}. \tag{1}$$

For the supported edge of a rectilinear plate the boundary conditions can be written as

$$w = 0, \quad \nabla_1^2 w = 0. \tag{2}$$

We can re-write (1) as

$$\nabla_1^2\left[\nabla_1^2 w - \tfrac{1}{4}k(x^2+y^2)\right] = 0. \tag{3}$$

Let

$$\chi = \nabla_1^2 w - \tfrac{1}{4}k(x^2+y^2), \tag{3.1}$$

then χ is such a harmonic function that over the boundary

$$\chi = -\tfrac{1}{4}k(x^2+y^2). \tag{3.2}$$

The determination of χ therefore reduces to finding the corresponding torsion function. Having known χ we can deduce w bearing in mind that the general bi-harmonic solution is of the form

$$\bar{z}\phi(z)+z\bar{\phi}(z)+\psi(z)+\bar{\psi}(z) \tag{3.3}$$

where ϕ and ψ are harmonic functions, $z = x+iy$ and bars denote the conjugate complex values. We need the following relations satisfied by a harmonic function

$$\nabla_1{}^2(x\phi) = 2\frac{\partial\phi}{\partial x}, \quad \nabla_1{}^2(y\phi) = 2\frac{\partial\phi}{\partial y}. \tag{3.4}$$

RECTANGULAR PLATE

To begin with we take the known case of a rectangular plate whose edges are $x = \pm a$, $y = \pm b$.

The torsion function being known we get

$$\nabla_1{}^2 w = \tfrac{1}{4}k(x^2-a^2) + a^2k(2/\pi)^3 \sum_0^\infty \frac{(-1)^n}{(2n+1)^3}\cdot\frac{\cos \alpha_n x/a \cosh \alpha_n y/a}{\cosh \alpha_n b/a}$$

$$+ \tfrac{1}{4}k(y^2-b^2) + b^2k(2/\pi)^3 \sum_0^\infty \frac{(-1)^n}{(2n+1)^3}\cdot\frac{\cos \alpha_n y/b \cosh \alpha_n x/b}{\cosh \alpha_n a/b}, \tag{4}$$

where $\alpha_n = \tfrac{1}{2}(2n+1)\pi$.

To determine w we can add a harmonic function to a particular integral of (4) such that $w = 0$ over the edges. A particular integral of (4) in obtained, by using (3 4), as

$$w_0 = \tfrac{1}{8}k[x^2y^2 + b^2(a^2-x^2) - a^2y^2] + \tfrac{1}{2}a^3k(2/\pi)^4.y \sum_0^\infty \frac{(-1)^n}{(2n+1)^4}\cdot\frac{\cos \alpha_n x/a \sinh \alpha_n y/a}{\cosh \alpha_n b/a}$$

$$+ \tfrac{1}{2}b^3k(2/\pi)^4.x \sum_0^\infty \frac{(-1)^n}{(2n+1)^4}\cdot\frac{\cos \alpha_n y/b \sinh \alpha_n y/b}{\cosh \alpha_n a/b}. \tag{5.1}$$

Since $w = 0$ over $x = \pm a$ and $y = \pm b$, the harmonic function to be added to w_0 to get w is

$$-\tfrac{1}{2}a^3k(2/\pi)^4 \sum_0^\infty \frac{(-1)^n}{(2n+1)^4}\, b \tanh\frac{\alpha_n b}{a} \operatorname{sech}\frac{\alpha_n b}{a} \cos\frac{\alpha_n x}{a} \cosh\frac{\alpha_n y}{a}$$

$$-\tfrac{1}{2}b^3k(2/\pi)^4 \sum_0^\infty \frac{(-1)^n}{(2n+1)^4}\, a \tanh\frac{\alpha_n a}{b} \operatorname{sech}\frac{\alpha_n a}{b} \cos\frac{\alpha_n y}{b} \cosh\frac{\alpha_n x}{b}. \tag{5.2}$$

Adding (5.1) and (5.2) we get the result (Love, 1927).

EQUILATERAL TRIANGULAR PLATE

We take the sides of the plate as $y = a$, $y = \pm\sqrt{3}x$. Proceeding as above we find

$$\nabla_1{}^2 w = \tfrac{1}{4}k(x^2+y^2) + \tfrac{1}{2}k(x^2-y^2) + \tfrac{1}{4}(k/a)(y^3-3x^2y) = \tfrac{1}{4}(k/a)(y-a)(y^2-3x^2). \tag{6}$$

$$\begin{aligned} w &= \tfrac{1}{8}kx^2y^2 + \tfrac{1}{12}ky(3x^2y-y^3) + \tfrac{1}{8}(k/a)x(xy^3-x^3y) + \tfrac{3}{64}k(x^4-6x^2y^2+y^4) \\ &\qquad + \tfrac{1}{64}(k/a)(5x^4y-10x^2y^3+y^5) - \tfrac{1}{48}ka(3x^2y-y^3) \\ &= \tfrac{1}{192}(k/a)(a-y)(3x^2-y^2)(3x^2+3y^2-4ay). \end{aligned} \tag{7}$$

w_c, the deflection at the centre, where it is a maximum is given by (Seth, 1945)

$$w_c = \frac{1}{4.3^5}\,ka^4. \tag{8}$$

ISOSCELES RIGHT-ANGLED TRIANGULAR PLATE

We take the sides of the plate as $x = a$, $y = a$, $y+x = 0$. From the known torsion solution (Seth, 1934) we get

$$\begin{aligned}\nabla_1{}^2 w &= \tfrac{1}{4}k(x^2+y^2) - \tfrac{1}{2}ka(x+y) + \tfrac{1}{2}kxy \\ &\quad + a^2k(2/\pi)^3 \sum_0^\infty \frac{(-1)^n}{(2n+1)^3}\cdot\frac{\cos\alpha_n x/a \sinh\alpha_n y/a + \cos\alpha_n y/a \sinh\alpha_n x/a}{\sinh\alpha_n}\end{aligned} \tag{9}$$

$$\begin{aligned} w &= \tfrac{1}{96}k(x+y)^4 - \tfrac{1}{24}ak(x+y)^3 + \tfrac{1}{12}a^3k(x+y) \\ &\quad - \tfrac{1}{2}ka^3(2/\pi)^4 \sum_0^\infty \frac{(-1)^n}{(2n+1)^4 \sinh^2\alpha_n}\left[\{a\cosh\alpha_n \sinh\alpha_n y/a - y\sinh\alpha_n \cosh\alpha_n y/a\}\cos\alpha_n x/a\right. \\ &\qquad\qquad \left. + \{a\cosh\alpha_n \sinh\alpha_n x/a - x\sinh\alpha_n \cosh\alpha_n x/a\}\cos\alpha_n y/a\right] \\ &\quad - \tfrac{1}{2}ka^4(2/\pi)^5 \sum_0^\infty \frac{(-1)^n}{(2n+1)^5 \sinh\alpha_n}\left[\sinh\alpha_n y/a \cos\alpha_n x/a + \sinh\alpha_n x/a \cos\alpha_n y/a\right].\end{aligned} \tag{10}$$

With the help of the expansions

$$a^2 - y^2 = 4a^2(2/\pi)^3 \sum_0^\infty \frac{(-1)^n}{(2n+1)^3}\cos\alpha_n y/a, \tag{11.1}$$

$$\tfrac{1}{12}y^4 - \tfrac{1}{2}a^2y^2 + \tfrac{5}{12}a^4 = 4a^4(2/\pi)^5 \sum_0^\infty \frac{(-1)^n}{(2n+1)^5}\cos\alpha_n y/a, \tag{11.2}$$

we see that $\nabla_1{}^2 w$ and w both vanish over the edges.

The central displacement w_c is given by

$$\begin{aligned} w_c = \frac{11ka^4}{3^3} &- \tfrac{4}{3}ka^4(2/\pi)^4 \sum_0^\infty \frac{(-1)^n}{(2n+1)^4}\left(\frac{\sinh\alpha_n/3}{\sinh\alpha_n}\right)^2 \sinh 2\alpha_n/3 \cos\alpha_n/3 \\ &- ka^4(2/\pi)^5 \sum_0^\infty \frac{(-1)^n}{(2n+1)^5}\cdot\frac{\sinh\alpha_n/3}{\sinh\alpha_n}\cos\alpha_n/3.\end{aligned} \tag{12}$$

The infinite series are found to be rapidly convergent and we get

$$w_c = 0.0103ka^4 \text{ (approx.)}. \tag{12.1}$$

For a square plate of side $2a$ we know that

$$w_c = 0.0325ka^4 \text{ (approx.)}. \tag{12.2}$$

The value of w_c given in (12.1) is the value of w at the centroid $(\frac{1}{3}a, \frac{1}{3}a)$. This is not the maximum value of w, which occurs near the point $(\frac{1}{2}a, \frac{1}{2}a)$, and its value, w_m, is found to be

$$w_m = .0166ka^4.$$

This is almost half the maximum displacement in the case of a square plate.

For $\sigma = 0.3$, the value of w_m obtained with the help of the double Fourier's series result due to Nadai (Timoshenko, 1940) is

$$w_m = (0.020)\,\frac{pa^4}{Eh^3}, \tag{12.4}$$

while we get a slightly higher value

$$w_m = (0.022)\,\frac{pa^4}{Eh^3}. \tag{12.5}$$

UNIFORMLY VARYING PRESSURE

Rectangular plate

In this case the harmonic function χ in (3.2) is of the third degree on the boundary of the plate, and hence, instead of the torsion function, we may have to take a function similar to the flexure function.

For a rectangular plate whose edges are $x = \pm a$, $y = \pm b$, we may write

$$\nabla_1{}^4 w = \frac{p}{D}\left(c_1\frac{x}{a} + c_2\frac{y}{b}\right), \tag{13}$$

from which we get

$$\begin{aligned}\nabla_1{}^2 w = {}& \tfrac{1}{2}c_1\frac{kx}{a}(y^2-b^2) + \tfrac{1}{2}c_2\frac{ky}{b}(x^2-a^2)\\ &+2k(2/\pi)^3\sum_0^\infty \frac{(-1)^n}{(2n+1)^3}\,[c_1b^2 \operatorname{cosech} \alpha_n a/b \sinh \alpha_n x/b \cos \alpha_n y/b\\ &\qquad + c_2a^2 \operatorname{cosech} \alpha_n b/a \sinh \alpha_n y/a \cos \alpha_n x/a],\end{aligned} \tag{14}$$

$$\begin{aligned}w = {}& \tfrac{1}{2}c_1\frac{kx}{a}\left(\tfrac{1}{12}y^4 - \tfrac{1}{2}b^2y^2 + \tfrac{5}{12}b^4\right) + \tfrac{1}{2}c_2\frac{ky}{b}\left(\tfrac{1}{12}x^4 - \tfrac{1}{2}a^2x^2 + \tfrac{5}{12}a^4\right)\\ &+k(2/\pi)^4\sum_0^\infty \frac{(-1)^n}{(2n+1)^4}\,[c_1b^3X_n \operatorname{cosech}^2 \alpha_n a/b \cos \alpha_n y/b + c_2a^3Y_n \operatorname{cosech}^2 \alpha_n b/a \cos \alpha_n x/a]\\ &-2k(2/\pi)^5\sum_0^\infty \frac{(-1)^n}{(2n+1)^5}\,[c_1b^4 \operatorname{cosech} \alpha_n a/b \sinh \alpha_n x/b \cos \alpha_n y/b\\ &\qquad + c_2a^4 \operatorname{cosech} \alpha_n b/a \sinh \alpha_n y/a \cos \alpha_n x/b],\end{aligned} \tag{15}$$

where

$$X_n = x \sinh \alpha_n a/b \cosh \alpha_n x/b - a \cosh \alpha_n a/b \sinh \alpha_n x/b,$$

and Y_n is obtained from X_n by interchanging x, y and a, b.

Right-angled isosceles triangular plate

In this case let

$$\nabla_1{}^4 w = k(x+y),$$

and we get

$$\nabla_1^2 w = \tfrac{1}{2}kx(y^2-a^2)+\tfrac{1}{2}ky(x^2-a^2)$$

$$+2a^3k(2/\pi)^3\sum_0^\infty \frac{(-1)^n}{(2n+1)^3}\cdot\frac{\sinh \alpha_n y/a \cos \alpha_n x/a+\sinh \alpha_n x/a \cos \alpha_n y/a}{\sinh \alpha_n}, \tag{16}$$

$$w = \tfrac{1}{2}kx(\tfrac{1}{12}y^4-\tfrac{1}{2}a^2y^2+\tfrac{5}{12}a^4)+\tfrac{1}{2}ky(\tfrac{1}{12}x^4-\tfrac{1}{2}a^2x^2+\tfrac{5}{12}a^4)$$

$$+a^4k(2/\pi)^4\sum_0^\infty L_{n,x,y}-2a^5k(2/\pi)^5\sum_0^\infty M_{n,x,y}, \tag{17}$$

where $L_{n,x,y}$ and $M_{n,x,y}$ are the general terms of the infinite series in (10).

General case

In the general case when the load is a given polynomial of any degree n, the corresponding value of χ can be determined from the following equations (Seth, 1935)

$$\frac{dz}{d\zeta} = A\prod_{r=1}^{m}(\zeta-\xi_r)^{\alpha_r-1}, \tag{18.1}$$

$$\frac{d\Omega_{n+1}}{d\zeta} = R(\zeta).\prod_{r=1}^{m}(\zeta-\xi_r)^{-(n+1)\alpha_r}. \tag{18.2}$$

In (18) ξ's are the points on the ξ-axis in the ζ-plane that correspond to the m angular points of the plate and $\pi\alpha_r$'s are the internal angles at these points. $R(\zeta)$ is a polynomial of degree $[(n+1)m-2(n+2)]$ with real coefficients, and Ω_{n+1} is given by

$$\Omega_{n+1} = \frac{d^{n+1}}{dz^{n+1}}(\phi+i\chi), \quad z = x+iy. \tag{19}$$

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ON BOUNDS OF SKEWNESS AND KURTOSIS

BY

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1. If $\bar{x}$ be the arithmetic mean of $n (\geq 3)$ real variates $x_1, x_2, \ldots, x_n$ and

$$\alpha_r = \{\sum (x_i - \bar{x})^r / n\} / \{\sum (x_i - \bar{x})^2 / n\}^{r/2},$$

then $\alpha_1 = 0$, $\alpha_2 = 1$, α_3 and α_4 respectively represent skewness and kurtosis. The result $\alpha_3 \leq (n-2)/\sqrt{(n-1)}$ is well-known, but a rigorous proof has been recently supplied by Wilkins (1944), who concludes his note with the remark: "It would probably be possible, although rather difficult, to derive an explicit bound for α_4 as a function of n by using the methods employed above for α_3." However, the result $\alpha_4 \leq (n-2) + 1/(n-1)$ is given without proof by Pearson (1916) in a foot-note on page 440 with the remark that it was given to him by Mr. G. N. Watson. Chakrabarti (1946) proves the two result by a different method.

Here I define α_{2p+1} and α_{2p+2} respectively as skewness and kurtosis of the pth order and obtain their upper and lower bounds. The two above bounds emerge as particular cases given by $p = 1$. Finally, I give a generalisation of the inequality $\alpha_4 \geq 1 + \alpha_3^2$.

2. The substitution

$$n(x_i - \bar{x})^2 = y_i^2 \sum (x_i - \bar{x})^2$$

gives

$$f_r(y) \equiv n\alpha_r = \sum y_i^r \text{ where } g(y) \equiv \sum y_i^2 - n = 0 \text{ and } h(y) \equiv \sum y_i = 0.$$

Let E_r be an extremum of α_r and e_{r-1} be the value of α_{r-1} corresponding to that set y_i which generates the extremum E_r. Let also

$$F_r(y) \equiv f_r(y) - \lambda g(y) - \mu h(y),$$

where λ, μ are Lagrange's multipliers. Then the extremizing y_i satisfy

$$\frac{\partial}{\partial y_i} F_r(y) \equiv r y_i^{r-1} - 2\lambda y_i - \mu = 0, \quad \sum y_i^2 = n, \ \sum y_i = 0$$

But

$$\sum \frac{\partial}{\partial y_i} F_r(y) = 0 \text{ and } \sum y_i \frac{\partial}{\partial y_i} F_r(y) = 0$$

give $\mu = r e_{r-1}$ and $2\lambda = r E_r$. Hence

$$y_i^{r-1} - E_r y_i - e_{r-1} = 0 \tag{1}$$

3. Bounds of skewness: Consider skewness of order p. (1) becomes

$$y_i^{2p} - E_{2p+1} y_i - e_{2p} = 0.$$

The last equation has exactly two real roots, η_1 and η_2; since $e_{2p} \geq 0$, $\eta_2 < 0$ if $\eta_1 > 0$. The set y_i thus contains η_1 repeated r_1 times and η_2 repeated r_2 times, where r_1, r_2 are positive integers whose sum is n. Elimination of η_1, η_2 from

$$r_1\eta_1 + r_2\eta_2 = 0, \quad r_1\eta_1^2 + r_2\eta_2^2 = n \quad \text{and} \quad nE_{2p+1} = r_1\eta_1^{2p+1} + r_2\eta_2^{2p+1}$$

leads to

$$E_{2p+1} = \frac{r_2^{2p} - r_1^{2p}}{n(r_1r_2)^{p-1/2}}$$

as the expression representing all extrema of α_{2p+1}. The numerater of this expression is maximised if $r_2 = n-1$ and $r_1 = 1$ while the denominator is minimised for the same partition of n. Hence the upper bound M_{2p+1} is given by

$$M_{2p+1} = \frac{(n-1)^{2p} - 1}{n(n-1)^{p-1/2}} \qquad \ldots\ (2)$$

and is attained when $\eta_1 = \sqrt{(n-1)}$ and $\eta_2 = -1/\sqrt{(n-1)}$. The lower bound m_{2p+1} is, of course, $-M_{2p+1}$. $p = 1$ gives $M_3 = (n-2)/\sqrt{(n-1)}$.

4. **Bounds of kurtosis:** $r = 2p+2$ gives kurtosis of order p and converts (1) into

$$y_i^{2p+1} - E_{2p+2}y_i - e_{2p+1} = 0$$

where $E_{2p+2} \geq 0$. If $e_{2p+1} > 0$, the equation has only one positive real root and exactly two real negative roots; if $e_{2p+1} < 0$, it has just two positive real roots and only one negative real root. In either case, there are just three real and distinct roots η_1, η_2, η_3. In the set y_i, therefore, η_1 is repeated r_1 times, η_2 repeated r_2 times and η_3 repeated r_3 times, where r_1, r_2, r_3 are non-negative integers whose sum is n.

I now regard r_1, r_2, r_3 as continuous variables and set

$$\theta \equiv r_1\eta_1^2 + r_2\eta_2^2 + r_3\eta_3^2 - n = 0,$$

$$\phi \equiv r_1\eta_1 + r_2\eta_2 + r_3\eta_3 = 0,$$

$$\psi \equiv r_2 + r_2 + r_3 - n = 0,$$

and

$$n\alpha_{2p+2} = r_1\eta_1^{2p+2} + r_2\eta_2^{2p+2} + r_3\eta_3^{2p+2}.$$

Let

$$\Theta \equiv n\alpha_{2p+2} - \lambda\theta - \mu\phi - \nu\psi,$$

where λ, μ, ν are Lagrange's multipliers. The extremising sets η_i, r_i satisfy

$$\frac{\partial\Theta}{\partial r_i} = 0, \quad \frac{\partial\Theta}{\partial\eta_i} = 0, \quad \theta = 0, \quad \phi = 0, \quad \psi = 0.$$

$$\sum \eta_i \frac{\partial\Theta}{\partial\eta_i} = 0 \text{ gives } \lambda = (p+1)E_{2p+2},$$

$$\sum \frac{\partial\Theta}{\partial\eta_i} = 0 \text{ gives } \mu = (2p+2)e_{2p+1},$$

and

$$\sum r_i \frac{\partial\Theta}{\partial r_i} = 0 \text{ gives } \nu = -E_{2p+2}.$$

Hence the nine extremising conditions reduce to

$$\eta_i^{2p+2} - E_{2p+2}(p+1)\eta_i^2 - 2e_{2p+1}(p+1)\eta_i + pE_{2p+2} = 0,$$

$$\eta_i^{2p+1} - E_{2p+2}\eta_i - e_{2p+1} = 0, \text{ and } \theta = 0, \quad \phi = 0, \quad \psi = 0.$$

From the first six conditions follow immediately the three conditions

$$\eta_i^2 + \frac{(2p+1)e_{2p+1}}{p\,E_{2p+2}} - 1 = 0$$

But η_1, η_2, η_3 cannot all be distinct if each satisfies the same quadratic. Hence only two of the three roots suffice to extremise the kurtosis. Without loss of generality, therefore,

$$r_3 = 0 \quad \text{and} \quad \eta_1\eta_2 = -1$$

Hence

$$r_1 + r_2 = n, \quad r_1 > 0,\ r_2 > 0,$$

$$r_1\eta_1 + r_2\eta_2 = 0, \quad \eta_1\eta_2 = -1,$$

$$r_1\eta_1^2 + r_2\eta_2^2 = n, \quad \text{and} \quad n\alpha_{2p+2} = r_1\eta_1^{2p+2} + r_2\eta_2^{2p+2}$$

Elimination of η_1, η_2 leads to

$$E_{2p+2} = \frac{r_1^{2p+1} + r_2^{2p+1}}{n(r_1r_2)^p} \quad \text{with } r_1 + r_2 = n,\ r_1 > 0,\ r_2 > 0$$

Now

$$\frac{d}{dr_1}E_{2p+2} = \frac{(2p+1)r_1r_2(r_1^{2p} - r_2^{2p}) + p(r_1 - r_2)(r_1^{2p+1} + r_2^{2p+1})}{n(r_1r_2)^{p+1}}$$

which is $<$ or >0 according as $r_2 >$ or $< r_1$, or $r_1 <$ or $> n/2$. Thus, E_{2p+2} decreases as r_1 increases from $r_1 = 1$ reaching a minimum at $r_1 = n/2$ and then increases as r_1 increases to $(n-1)$. Hence the upper bound will be given by either $r_1 = 1$ or $r_1 = n-1$. In either case, the value is the same. Hence

$$M_{2p+2} = \frac{(n-1)^{2p+1} + 1}{n(n-1)^p}, \qquad \dots \quad (3)$$

and is attained if $\eta_1 = \sqrt{(n-1)}$ and $\eta_2 = -1/\sqrt{(n-1)}$ with $r_1 = 1$. $p = 1$ gives $M_4 = (n^2 - 3n + 3)/(n-1)$, which is Watson's value.

The lower bound will be given by $r_1 = r_2 = n/2$ if n is even, and by $r_1 = (n-1)/2$ or $(n+1)/2$ if n is odd. In the first case, the lower bound m_{2p+2} is 1 and is attained if $\eta_1 = 1$, $\eta_2 = -1$; in the second case, the lower bound is

$$m_{2p+2} = \frac{(n+1)^{2p+1} + (n-1)^{2p+1}}{2n(n^2-1)^p}, \qquad \dots \quad (4)$$

and is attained when

$$\eta_1 = -\sqrt{\left(\frac{n+1}{n-1}\right)}, \quad \eta_2 = -\sqrt{\left(\frac{n-1}{n+1}\right)}, \; r_1 = \frac{n-1}{2}$$

If $p = 1$, $m_4 = \dfrac{n^2+3}{n^2-1}$, so that

$$\frac{n^2+3}{n^2-1} \leq \alpha_4 \leq \frac{n^2-3n+3}{n-1}, \quad \text{if } n \text{ is odd.}$$

and

$$1 \leq \alpha_4 \leq \frac{n^2-3n+3}{n-1}, \quad \text{if } n \text{ is even.}$$

5. Pearson's inequality $\alpha_4 \geq 1 + \alpha_3^2$ is readily amenable to generalisation by Wilkins' Method. Let

$$G(a_0, a_1, \ldots, a_k) \equiv (1/n)\sum(a_0 + a_1 y_i + \ldots + a_k y_i^k)^2 = \sum\sum \alpha_{i+j} a_i a_j.$$

G is positive semi-definite, so that for all integers $r \geq 2$

$$\Delta_r \equiv \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \ldots & \alpha_r \\ \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{r+1} \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \alpha_r & \alpha_{r+1} & \alpha_{r+2} & \ldots & \alpha_{2r} \end{vmatrix} \geq 0,$$

where $\alpha_0 = \alpha_2 = 1$ and $\alpha_1 = 0$

If $r = 2$, $\Delta_2 \geq 0$ reduces to Pearson's inequality.

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A TYPE OF SOLUTIONS OF EINSTEIN'S GRAVITATIONAL EQUATIONS

By

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1. The Gravitational equations of Einstein have been solved for a very limited number of cases, including the field of a homogeneous material sphere in which the internal stresses are isotropic, *i.e.*, the stress-components represented by T_1^1, T_2^2, T_3^3 are equal (Schwarzschild, 1916; De Sitter, 1916; Einstein, 1917; Ghosh, 1926, 1928; Sen, 1924). But there are cases of spherical body under symmetrical strain, where the radial and transverse stresses are not equal. A simple case in which the radial and transverse stresses are linearly related has been worked out in this paper. The original assumption and the deduction of the differential equation which yields the space-time metric was obtained in a previous paper by the author, published in the Indian Physico-Mathematical Journal, Vol. III. As the journal is now defunct, the present paper starts with the recapitulation of the preliminary steps.

2. Assuming the gravitational equations in the form

$$K_{pq}-\tfrac{1}{2}g_{pq}K+\beta g_{pq} = -8\pi T_{pq}, \tag{1}$$

where β is the cosmological constant, and T_{pq} is the material-energy-tensor, and the radially symmetric field is given by the metric

$$ds^2 = -(e^{-\lambda}dr^2+r^2d\theta^2+r^2 \sin^2\theta d\phi^2)+e^{\nu}dt^2, \tag{2}$$

the equations (1) give

$$-8\pi\left(T_1^1+\frac{\beta}{8\pi}\right) = e^{-\lambda}\left(\frac{\nu'}{r}-\frac{e^{\lambda}-1}{r^2}\right) \tag{3}$$

$$-8\pi\left(T_2^2+\frac{\beta}{8\pi}\right) = e^{-\lambda}\left\{\tfrac{1}{2}\nu''-\tfrac{1}{4}\lambda'\nu'+\tfrac{1}{4}\nu'^2+\tfrac{1}{2}\frac{\nu'+\lambda'}{r}\right\} \tag{4}$$

$$T_3^3 = T_2^2 \tag{5}$$

$$-8\pi\left(T_4^4+\frac{\beta}{8\pi}\right) = e^{-\lambda}\left(-\frac{\lambda'}{r}-\frac{e^{\lambda}-1}{r^2}\right) \tag{6}$$

It has been shown that (6) with $T_4^4 = \rho$, constant, gives

$$e^{-\lambda} = 1-ar^2-b/r,$$

where $a = \frac{1}{3}(8\pi\rho+\beta)$.

As the field extends to the origin, b is to be taken as zero and we have

$$e^{-\lambda} = 1-ar^2. \tag{7}$$

Assuming $T_2{}^2 = mT_1{}^1 + n$. it has been shown that (3) and (4) give

$$(r - ar^2)\frac{d^2z}{dr^2} - \{(2m-1) - 2(m-1)ar^2\}\frac{dz}{dr} + r\{(m-1)a - N\} = 0, \tag{8}$$

where

$$z = e^{\frac{1}{2}\nu} \quad \text{and} \quad N \equiv (m-1)\beta - 8\pi n. \tag{9}$$

3. It appears from (7) that the instantaneous ($dt = 0$) spatial geometry of the space depends on the density, but is independent of the distribution of stress.

Since $ar^2 < 1$ and $a = \frac{1}{3}(8\pi\rho + \beta)$, it follows that r must be less than $\sqrt{\{3/(8\pi\rho + \beta)\}}$. This sets an upper limit to the size of a sphere of density ρ.

4. If we require that the solution of (8) should not be infinite at the origin, we assume

$$z = C_0 + C_1 r + C_2 r^2 + \cdots + C_k r^k + \cdots \tag{10}$$

Substituting in (8), we get

$$C_{k+1}(k+1)(k+1-2m) = C_{k-1}.a\{(k-1)(k-2m) - N'\}, \tag{11}$$

where

$$aN' = a(m-1) - N. \tag{12}$$

Hence

$$z = c_0\phi(r) + c_1\psi(r), \tag{13}$$

where

$$\phi(r) = 1 - \frac{N'}{2(2-2m)}ar^2 - \frac{N'\{2(3-2m) - N'\}}{2.4.(2-2m)(4-2m)}a^2r^4 - \cdots \tag{14}$$

$$\frac{\psi(r)}{r} = 1 + \frac{(2-2m) - N'}{3(3-2m)}ar^2 + \frac{\{(2-2m) - N'\}\{3(4-2m) - N'\}}{3.5.(3-2m)(5-2m)}a^2r^4 + \cdots \tag{15}$$

From (11), we have

$$\text{Lt}\,\frac{C_{k+1}}{C_{k-1}} = \text{Lt}\,\frac{\{(k-1)(k-2m) - N'\}}{(k+1)(k+1-2m)}ar^2 = ar^2.$$

Since $ar^2 < 1$, the series (13) and (14) are convergent for all possible values of r.

5. From (3), we get

$$-8\pi\left(T_1{}^1 + \frac{\beta}{8\pi}\right) = (1 - ar^2)\left\{\frac{1}{r}.\frac{2(c_0\phi' + c_1\psi')}{c_0\phi + c_1\psi} + \frac{1}{r^2}\right\} - \frac{1}{r^2},$$

giving

$$-8\pi T_1{}^1 = (1 - ar^2)\left\{\frac{2(c_0\phi' + c_1\psi')}{c_0\phi + c_1\psi}\right\} - (a + \beta).$$

If the sphere has a radius r_0 and the surface of the sphere is free, we have $T_1{}^1 = 0$ where $r = r_0$, This gives

$$2(1 - ar_0{}^2)(c_0\phi_0' + c_1\psi_0') - (a + \beta)(c_0\phi_0 + c_1\psi_0)r_0 = 0, \tag{16}$$

where ϕ_0 is the value of $\phi(r)$ at $r = r_0$, etc.

At $r = r_0$, we arrive at empty space of De Sitter. Hence

$$e^{\nu} = 1 - \frac{2M}{r_0} - \tfrac{1}{3}\beta r_0^2,$$

M being a constant, identifiable with the mass of the sphere.

Hence at $r = r_0$, we have

$$c_0\phi_0 + c_1\psi_0 = k_0, \tag{17}$$

where ϕ_0 is the value of $\phi(r)$ at $r = r_0$, ψ_0 is the value of $\psi(r)$ at $r = r_0$, and

$$k_0 = \sqrt{\left(1 - \frac{2M}{r_0} - \tfrac{1}{3}\beta r_0^2\right)}.$$

Equations (16) and (17) determine the constants c_0 and c_1. In fact, we have

$$c_0 = \frac{(a+\beta)r_0\psi_0 - 2\psi_0'(1-ar_0^2)}{2(1-ar_0^2)(\psi_0\phi_0' - \phi_0\psi_0')}k_0, \tag{18}$$

$$c_1 = \frac{2\phi_0'(1-ar_0^2) - (a+\beta)r_0\phi_0}{2(1-ar_0^2)(\psi_0\phi_0' - \phi_0\psi_0')}k_0. \tag{19}$$

Thus (7), (13), (18) and (19) give the complete solution of the problem.

6. We may now consider a particular case in which the radial and transverse stresses are proportional. We have in the case $T_2^2 = mT_1^1$, so that $n = 0$. Since the natural curvature of space is very small compared with the curvature produced by the matter composing the sphere, we may further put $\beta = 0$.

We then have

$$N = 0 \quad \text{and} \quad N' = m-1. \tag{20}$$

The equation (8) becomes

$$(r-ar^2)\frac{d^2z}{dr^2} - \{(2m-1) - 2(m-1)ar^2\}\frac{dz}{dr} + r(m-1)az = 0. \tag{21}$$

The solution is then

$$z = c_0\phi(r) + c_1\psi(r), \tag{22}$$

where

$$\phi(r) = 1 + \tfrac{1}{4}ar^2 + \frac{7-5m}{32(2-m)}a^2r^4 + \cdots \tag{23}$$

and

$$\frac{\psi(r)}{r} = 1 + \frac{1-m}{3-2m}ar^2 + \frac{(1-m)(13-7m)}{5(3-2m)(5-2m)}a^2r^4 + \cdots \tag{24}$$

HUGHLI COLLEGE,
WEST BENGAL.

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CALCUTTA MATHEMATICAL SOCIETY

Report of the Council for the year 1947 to the Annual General Meeting of the Society

The Council of the Calcutta Mathematical Society has the pleasure to submit the following report on the general concerns of the Society for the year 1947, as required by the provisions of Rule 25.

The Council: The Council of the Society for the year 1947 consisting of the officers and members elected at the last Annual General meeting together with the Editorial Secretary, was constituted as follows:—

President

Prof. N. M. Basu

Vice-Presidents

Prof. A. C. Banerjee
Prof. N. R. Sen
Prof. M. R. Siddiqi.
Dr. R. N. Sen
Dr. S. S. Pillai

Treasurer

Mr. S. C. Ghosh

Secretary

Mr. U. R. Burman

Editorial Secretary

Mr. S. Gupta (Upto August, 1947)
Mr. P. K. Ghosh (from September, 1947)

Other members of the Council

Dr. B. R. Seth,
Prof. S. N. Bose,
Prof. B. B. Sen,
Mr. A. C. Choudhury,
Prof. F. W. Levi,
Prof. C. V. H. Rao,
Dr. N. G. Shabde,
Mr. B. C. Chatterjee,
Dr. P. N. Dasgupta,
Dr. Gorakh Prosad,
Dr. B. S. Ray,
Mr. P. K. Ghosh
(Assistant Secretary)

General: It has not been possible to resume yet the various activities of the Society which have so far been conducted with the usual wartime arrangements. The Council is however glad to report that it was possible to arrange a public lecture by Prof. L. J. Mordell, Sadlerian Professor of Mathematics in the University of Cambridge (England), during the Professor's visit to this country to attend the 34th session of the Indian Science Congress held at Delhi in January, 1947. The Council offers its most sincere thanks to Prof. Mordell for the very interesting lecture he delivered on the works of the Indian Mathematician Ramanujan.

The Society was very much unfortunate in losing the services of the Editorial Secretary Mr. S. Gupta, who had to leave us in September, 1947, to take up an appointment in the Tata Institute for Fundamental Research, Bombay. The Council takes this opportunity to offer him its grateful thanks for the valued services he rendered to the Society for the past few years.

Membership : The Council records with profound regret the death of two distinguished Honorary members of the Society, Prof. G. H. Hardy, of the Cambridge University (England) and Prof. C. W. Oseen of the Uppsala University (Sweden). The Council also learns with a deep sorrow of the death of Prof. C. V. Hanumanta Rao, whose services as a former Vice-president and a member of the Board of Editors of the Bulletin, the Council recalls with grateful appreciation.

During the year under review 4 new members were elected.

Meetings During 1947 : The Council held five meetings during the year and there were three ordinary general meetings which were devoted to the reading of original papers communicated to the Society for publication in the Bulletin.

Publications : Two numbers of the Bulletin were published during the year 1947, *viz.*, No. 4 of Vol. 38, and No. 1 of Vol. 39. The Society was faced with some unforeseen difficulties in the matter of publications and inspite of best efforts it was not possible to bring out more issues of the Bulletin during the year under review.

The Society received with grateful thanks a sum of Rs. 650/- from the Rockefeller Foundation Grant, distributed through the National Institute of Sciences, India. The Council has the pleasure to report that this grant has largely contributed towards the improvement of the Bulletin.

The authorities of the Calcutta University have continued to lend their usual magnanimous services in printing the Bulletin free of charge, and the Officers and members of the staff of the University Press have given their every sympathetic and active co-operation in bringing out the Bulletin in these difficult times. The Council takes this opportunity of offering them its very sincere thanks.

Exchange of Publications : The distribution of Bulletins to countries with which communication was impracticable during the war period has been carried on with increasing facilities, and some new exchange relations have also been established during the year under review.

Finance : The annual accounts of the Society for the year 1947 have been presented to the Council in the usual form by the auditors Mr. L. K. Ray and Mr. N. L. Ghosh, who deserve the Council's grateful thanks for their honorary services. The Council desires to point out that the decline noticed on the receipts side under the head Interest is partly due to the diminished rate of interest reported in the previous year, and partly due to a deduction on account of income-tax, which latter will however be refunded to the Society, when the necessary Income-tax exemption certificate is obtained. The financial position of the Society as a whole remains much the same as in the past few years.

CALCUTTA MATHEMATICAL SOCIETY

RECEIPTS AND DISBURSEMENTS ACCOUNTS OF THE CALCUTTA MATHEMATICAL SOCIETY FOR THE YEAR ENDING 31ST DECEMBER, 1947.

Receipts				Rs. as. p.
1. Opening balance				
(a) Cash with Secretary	....		2 1 9	
(b) Balance at Bank				
(i) Imperial Bank of India		1194 10 6		
(ii) Do. (K.K.G. P. Fund)		283 13 3		
(iii) Bengal Central Bank		305 15 0		
(iv) Bengal Central Bank (By deposit of cash left with Secretary on 31.12.46)		20 0 0		
(v) P. O. Savings Bank		578 5 0		
			2382 11 9	
(c) G. P. Notes (General Fund) (Face value Rs. 6,000)	...		5663 11 6	
(d) G. P. Notes (K.K.G.P. Fund) (Face value Rs. 2,000)	...		1937 7 9	
				9986 0 9
2. Membership Subscriptions	...		886 8 0	
3. Admission Fees	...		40 0 0	
4. Sale Proceeds	...		813 8 6	
5. Rockefeller Foundation Grant	...		550 0 0	
				2290 0 6
6. Interest				
(a) G. P. Notes (General Fund)	...		105 2 0	
(b) Do. (K. K. G. P. Fund)	...		34 2 5	
(c) P. O. Savings Bank	...		8 11 0	
				147 15 5
			TOTAL Rs.	12,424 0 8

Disbursements				Rs. as. p.
1. Establishment	...		785 0 0	
2. Meetings	...		63 3 6	
3. Books and Journals (including binding charges)			265 4 0	
4. Bulletins				
(a) Papers, Blocks and Types		285 3 0		
(b) Postage		122 6 0		
			407 9 0	
5. Printing and Stationery	...		23 12 0	
6. Postage		52 2 9		
Less stamps in hand		2 0 9		
			50 2 0	
7. Bank Charges	...		7 13 0	
8. Miscellaneous (including conveyance charges)...			45 3 6	
				1647 15 0
Closing Balance				
(a) With Secretary				
(i) In cash		17 5 0		
(ii) In stamps		2 0 9		
			19 5 9	
(b) Balance at Bank				
(i) Imperial Bank of India		1299 7 6		
(ii) Do. (K. K. G. P. Fund)		317 11 8		
(iii) Bengal Central Bank		951 5 6		
(iv) P. O. Savings Bank		587 0 0		
			3155 8 8	
(c) G. P. Notes (General Fund) (Face value Rs. 6,000)	...		5663 11 6	
(d) G. P. Notes (K. K. G. P. Fund) (Face value Rs. 2,000)	...		1937 7 9	
				10776 1 8
			TOTAL ...	12,424 0 8

To
The Members of the Calcutta Mathematical Society.

We have examined the above Balance Sheet with the Books and Vouchers relating thereto and certify it to be correctly drawn up therefrom and in accordance with the information and explanations given to us.

L. K. RAY
N. GHOSE *Auditors*

ON THE BENDING OF AN ELLIPTIC PLATE UNDER CERTAIN DISTRIBUTION OF LOAD—(III)

By

H. M. Sengupta

(*Communicated by Prof N. M. Basu—Received March 1, 1948*)

In a previous paper (1948) the author has determined the deflexion of the central plane of a thin elliptic plate, made of isotropic elastic material, under the action of a load of weight W concentrated at any point on the line joining the foci, the edge of the plate being clamped. The elliptic co-ordinates of the point where the load has been placed being $(0, \beta)$ $(0 \leqq \beta \leqq \pi)$, the deflexion is given by formula (7) of that paper read together with the formulæ (9), (10), (11), . . ., (19) of the same paper. The cartesian co-ordinates of the loaded point are $x' = c \cos \beta$, $y' = 0$. The deflexion at any point (x, y) of the above plate given by formula (7) of the paper mentioned above may be written as

$$w = -\frac{W}{16\pi D}\{(x-x')^2+y^2\}[\log\{(x-x')^2+y^2\}-2\log\tfrac{1}{2}c]+A'_0+A_0\cosh 2\xi$$
$$+(A'_1\cosh\xi+A_1\cosh 3\xi)\cos\eta+\sum_{n=2}^{\infty}\{A_{n-2}\cosh(n-2)\xi$$
$$+A'_n\cosh n\xi+A_n\cosh(n+2)\xi\}\cos n\eta \quad (1)$$

The positive direction of the z-axis has been taken in the opposite direction towards which the load acts and $D = \dfrac{2}{3}\dfrac{Eh'^3}{1-\sigma^2}$ where $2h'$ is the thickness of the plate and E and σ are the Young's Modulus and Poisson's ratio respectively of the material of the elastic plate.

The above defines a continuous function biharmonic throughout the plate excepting at the point $(x=x', y=0)$, *i.e.*, where the load has been placed. The function is not defined for $(x', 0)$. It, however, tends to a finite definite limit as (x, y) tends to $(x', 0)$.

We now propose to determine the deflexion when a load of equivalent weight is spread uniformly on any part of the line joining the foci.

Let the cartesian co-ordinates of the end points of the part of the major axis exposed to the pressure be denoted by $(X_1, 0)$ and $(X_2, 0)$ where $-c \leqq X_1 < X_2 \leqq c$. Let the elliptic co-ordinates of the above points be $(0, \beta_1)$ and $(0, \beta_2)$ respectively. Then $X_1 = c\cos\beta_1$ and $X_2 = c\cos\beta_2$ and $0 \leqq \beta_1 < \beta_2 \leqq \pi$. The part of the major axis exposed to the pressure has the length $X_1 - X_2 = c(\cos\beta_1 - \cos\beta_2)$.

Since the total load is of weight W and the load is uniform, the load per unit length is of weight $W/(X_1 - X_2) = W/c(\cos\beta_1 - \cos\beta_2)$ and the load on an element of length $\delta x'$ is of weight $W\delta x'/(X_1 - X_2) = W\delta x'/c(\cos\beta_1 - \cos\beta_2)$.

The deflexion on account of a load of weight $W\delta x'/(X_1 - X_2)$ placed at the point $(x', 0)$ along the major axis is obtained by replacing W by $W\delta x'/(X_1 - X_2)$ in the formula

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given above and the deflexion when the load is spread uniformly from X_2 to X_1 is given by the integral

$$-\frac{W}{16\pi D(X_1-X_2)}\int_{X_2}^{X_1}\{(x-x')^2+y^2\}[\log\{(x-x')^2+y^2\}-2\log\tfrac{1}{2}c]dx'$$
$$+\int_{X_2}^{X_1}[\bar{A}'_0+\bar{A}_0\cosh 2\xi+(\bar{A}'_1\cosh\xi+\bar{A}_1\cosh 3\xi)\cos\eta$$
$$+\sum_{n=2}^{\infty}\{\bar{A}_{n-2}\cosh(n-2)\xi+\bar{A}'_n\cosh n\xi+\bar{A}_n\cosh(n+2)\xi\}\cos n\eta]dx'$$

where $\bar{A}_n$ stands for the modified expression for A_n when W appearing in A_n has been replaced by $W/(X_1-X_2)$.

We however start with the integral

$$U=-\frac{W}{16\pi D(X_1-X_2)}\int_{X_2}^{X_1}\{(x-x')^2+y^2\}[\log\{(x-x')^2+y^2\}-2\log\tfrac{1}{2}c]dx' \qquad (2)$$

where the parameters x, y stand for the co-ordinates of any point on the mid-plane of the elliptic plate while x' is the x-coordinate of any point lying on the part of the major axis whose end points have coordinates $(X_1, 0)$ and $X_2, 0)$ respectively.

So long as (x, y) lies outside the line joining the points $(X_2, 0)$ and $(X_1, 0)$ the integral (2) has a definite finite value as the integrand in that case is continuous and bounded throughout $X_2 \leqq x' \leqq X_1$.

It represents a finite and continuous function in any closed region that does not include any point of the line obtained by joining $(X_2, 0)$ and $(X_1, 0)$. At every point of the above region we can easily show that

$$\nabla_1^4\int_{X_2}^{X_1}\{(x-x')^2+y^2\}[\log\{(x-x')^2+y^2\}-2\log\tfrac{1}{2}c]dx'=0.$$

The integral therefore represents a continuous function biharmonic throughout the plate excepting at points that lie on the line joining $(X_2, 0)$ and $(X_1, 0)$. Partial derivatives of all orders with respect to x and y and as also mixed derivatives are finite and continuous everywhere throughout the plate (ensuring finiteness and continuity of the contribution to the stress resultants and stress-couples arising out of this integral) excepting at the points that lie on the line joining $(X_2, 0)$ and $(X_1, 0)$.

At points $(x, 0)$ that lie on the line (X_2, X_1) the integral reduces to

$$-\frac{W}{16\pi D(X_1-X_2)}\int_{X_2}^{X_1}(x-x')^2\{\log(x-x')^2-2\log\tfrac{1}{2}c\}dx'.$$

The integrand is defined and continuous in the intervals $X_2 \leqq x' < x$ and $x < x' \leqq X_1$. It has no value for $x'=x$. The function is however bounded in the range of integration. In fact the integrand tends to the limit c as $x' \rightarrow x$ either from the left or from the right. The R-integral therefore exists and its value is

$$U=\frac{W}{48\pi D(X_1-X_2)}[\{(x-X_1)^3\log(x-X_1)^2-(x-X_2)^3\log(x-X_2)^2\}$$
$$-\tfrac{2}{3}\{(x-X_1)^3-(x-X_2)^3\}\cdot(1+3\log\tfrac{1}{2}c) \qquad (3)$$

For $x = X_1$, $y = 0$

$$U = -\frac{W}{48\pi D}[(X_1 - X_2)^3 \log(X_1 - X_2)^2 - \tfrac{2}{3}(X_1 - X_2)^2(1 + 3 \log \tfrac{1}{2}c)]. \tag{4}$$

It has exactly the same value for the point

$$x = X_2, \qquad y = 0.$$

For (x, y) such that $y \neq 0$, we may easily show that

$$U = \frac{W}{48\pi D(X_1 - X_2)}\Big[[(x - X_1)^3 \log R_1^2 - (x - X_2)^3 \log R_2^2 - \tfrac{2}{3}\{(x - X_1)^3 - (x - X_2)^3\}\cdot(1 + 3 \log \tfrac{1}{2}c)$$
$$+ 3y^2\{(x - X_1)\log R_1^2 - (x - X_2)\log R_2^2\} + 4y^2(X_1 - X_2)(1 + \tfrac{8}{3} \log \tfrac{1}{2}c)$$
$$+ 4y^3\left\{\tan^{-1}\frac{x - X_1}{y} - \tan^{-1}\frac{x - X_2}{y}\right\}\Big]. \tag{5}$$

where $R_1^2 = (x - X_1)^2 + y^2$ and $R_2^2 = (x - X_2)^2 + y^2$, while for $y = 0$ but x not lying on the line $< X_2, X_1 >$

$$U = \frac{W}{48\pi D(X_1 - X_2)}[\{(x - X_1)^3 \log(x - X_1)^2 - (x - X_2)^3 \log(x - X_2)^2\}$$
$$- \tfrac{2}{3}\{(x - X_1)^3 - (x - X_2)^3\}(1 + 3 \log \tfrac{1}{2}c)] \tag{6}$$

We thus see that $U(x, y)$ is continuous throughout the plate.

We can easily show that for $y \neq 0$

$$U'_x(x, y) = \frac{W}{16\pi D(X_1 - X_2)}[(R_1^2 \log R_1^2 - R_2^2 \log R_2^2) - 2\{(x - X_1)^2 - (x - X_2)^2\}\log \tfrac{1}{2}c]$$

$$U'_x(x, 0) = \frac{W}{16\pi D(X_1 - X_2)}[\{(x - X_1)^2 \log(x - X_1)^2 - (x - X_2)^2 \log(x - X_2)^2\}$$
$$- 2\{(x - X_1)^2 - (x - X_2)^2\}\log \tfrac{1}{2}c]$$

for $X_2 < x < X_1$.

$U'_x(x, 0)(x > X_1$ and $x < X_2)$ is also given by the same formula

$$U'_x(X_1, 0) = \frac{W}{16\pi D(X_1 - X_2)}[-(X_1 - X_2)^2 \log(X_1 - X_2)^2 + 2(X_1 - X_2)^2 \log \tfrac{1}{2}c]$$

and

$$U'_x(X_2, 0) = \frac{W}{16\pi D(X_1 - X_2)}[(X_1 - X_2)^2 \log(X_1 - X_2)^2 - 2(X_1 - X_2)^2 \log \tfrac{1}{2}c].$$

Thus the partial derivative of U with respect to x is continuous throughout the plate. We can show without difficulty that the first partial derivative of U with respect to y as well as all partial derivatives of the second order with respect to x and y are finite and continuous throughout the plate.

Thus contributions due to U to G and H the stress-couples that depend on derivatives upto the second order of the deflexion function are finite and continuous throughout the plate.

We can easily show that for all points $(x, y)(y \neq 0)$

$$\nabla_1^2 U = \frac{W}{24\pi D(X_1 - X_2)}\Big[6\{(x-X_1)\log R_1^2 - (x-X_2)\log R_2^2\} + (X_1 - X_2)(1 + 12\log \tfrac{1}{2}c) + 12y\Big\{\tan^{-1}\frac{x-X_1}{y} - \tan^{-1}\frac{x-X_2}{y}\Big\}\Big]$$

and for points on $y=0$, the corresponding value of $\nabla_1^2 U$ is the limit of the above function as (x, y) tends to appropriate point on the x-axis. Let N_U stand for the contribution to the normal shearing stress N due to U, across an arc Δ. Then across a line parallel to y-axis *i.e.*, $x =$ const.

$$N_{U,x} = -D\frac{\partial}{\partial x}\nabla_1^2 U$$

We use the symbol $N_{U,x}$ for the value of N_U across a line $x =$ const. So for any point of such a line for which $y \neq 0$

$$N_{U,x} = \frac{W}{4\pi(X_1 - X_2)}[\log R_1^2 - \log R_2^2]$$

Thus $N_{U,x}$ has an infinite discontinuity in the neighbourhood of the extremities of the strip $X_2 \leqq x \leqq X_1$. Across a line parallel to the x-axis, *i.e.*, across $y =$ const.

$$N_{U,y} = -D\frac{\partial}{\partial y}\nabla_1^2 U$$

where $N_{U,y}$ is used to denote the value of N_U across a line $y =$ const., for $y \neq 0$

$$N_{U,y} = -\frac{W}{2\pi(X_1 - X_2)}\left[\tan^{-1}\frac{x-X_1}{y} - \tan^{-1}\frac{x-X_2}{y}\right]$$

Hence for any point (x, y) such that $X_2 < x < X_1$

$$N_{U,y} = \frac{W}{2\pi(X_1 - X_2)}\left[\tan^{-1}\frac{X_1 - x}{y} + \tan^{-1}\frac{x-X_2}{y}\right]$$

tends to $\dfrac{W}{2(X_1 - X_2)}$ as y tends to $+0$. It tends to $-\dfrac{W}{2(X_1 - X_2)}$ as y tends to -0.

So there is a discontinuity in the value of $N_{U,y}$ in crossing from one side of the strip $X_2 < x < X_1$; $y = 0$, to the other.

If, however, $x > X_1 > X_2$

$$N_{U,y} = -\frac{W}{2\pi(X_1 - X_2)}\left[\tan^{-1}\frac{x-X_1}{y} - \tan^{-1}\frac{x-X_2}{y}\right]$$

So $N_{U,y} \to 0$ as $y \to 0$ either through positive or through negative values. There is thus no discontinuity in the value of $N_{U,y}$ in crossing any part of the x-axis that lies to the right of the strip $X_2 \leqq x \leqq X_1$. The same is also true of the part that lies to the left of the strip $X_2 \leqq x \leqq X_1$.

We can show by direct integration that the value of $\int N_U ds$ taken along the edges of the rectangle $x_2 \leqq x \leqq x_1$; $-l \leqq y \leqq l$ where $-c \leqq X_2 < x_2 < x_1 < X_1 \leqq c$ and $l(>0)$ is small enough to ensure that the above rectangle lies completely within the ellipse is equal to

$$W(x_1 - x_2)/(X_1 - X_2) \tag{7}$$

While the value of the integral $\int N_U ds$ taken along the edges of the rectangle $d_2 \leqq x \leqq d_1$; $-l \leqq y \leqq l$ where $-a < d_2 < X_2 < d_1 < X_1 < a$ and $l(>0)$ is small enough to ensure that the second rectangle lies completely within the ellipse is equal to

$$W(d_1 - X_2)(X_1 - X_2). \tag{8}$$

Also the value of $\int N_U ds$ taken along the edges of the rectangle $e_2 \leqq x \leqq e_1$; $-l \leqq y \leqq l$ where $-a < X_2 < e_2 < X_1 < e_1 < a$ and $l(>0)$ is small enough to ensure that the above rectangle lies completely within the ellipse is equal to

$$W(X_1 - e_2)/(X_1 - X_2). \tag{9}$$

For the deflexion we now assume

$$w = U + w' \tag{10}$$

where

$$U = -\frac{W}{16\pi D(X_1 - X_2)} \int_{X_2}^{X_1} \{(x-x')^2 + y^2\}[\log\{(x-x')^2 + y^2\} - 2 \log \tfrac{1}{2}c]\,dx' \tag{11}$$

and

$$\begin{aligned} w' = {} & B'_0 + B_0 \cosh 2\xi + (B'_1 \cosh \xi + B_1 \cosh 3\xi)\cos \eta \\ & + \sum_{n=2}^{\infty} \{B_{n-2} \cosh(n-2)\xi + B'_n \cosh n\xi + B_n \cosh(n+2)\xi\}\cos n\eta \end{aligned} \tag{12}$$

We have now to adjust the co-efficients B'_n and B_n in such a manner that $w = 0$ and $\partial W/\partial \xi = 0$ hold at all points of the bounding ellipse $\xi = \alpha$.

Now for $\xi > 0$, *i.e.*, for (x, y) such that it does not lie on the line joining the foci we have

$$\begin{aligned} & -\frac{W}{16\pi D(X_1 - X_2)}\{(x-x')^2 + y^2\}[\log\{(x-x')^2 + y^2\} - 2 \log \tfrac{1}{2}c] = -\frac{W}{8\pi D(X_1 - X_2)} r^2(\log r - \log \tfrac{1}{2}c) \\ & = -\frac{Wc^2}{32\pi D(X_1 - X_2)}\Big[\{\xi(2 \cosh 2\xi + 2 + 2 \cos 2\beta) + 2(1 + e^{-2\xi}) + (2 + e^{-2\xi})\cos 2\beta\} \\ & \quad -\{8\xi \cosh \xi \cos \beta + 2\xi \cos \beta + e^{-\xi}(7 \cos \beta + \cos 3\beta) + e^{-3\xi}(\cos \beta - \tfrac{1}{3} \cos 3\beta)\}\cos \eta \\ & \quad +\{2\xi + (2 + \cos 2\beta) + e^{-2\xi}(1 + \tfrac{2}{3} \cos 2\beta - \tfrac{1}{3} \cos 4\beta) + e^{-4\xi}(-\tfrac{1}{3} \cos 2\beta + \tfrac{1}{6} \cos 4\beta)\}\cos 2\eta \\ & \quad + 2\sum_{n=3}^{\infty}\Big\{e^{-(n-2)\xi}\Big(-\frac{\cos \overline{n-2}\,\beta}{n-2} + \frac{2 \cos \beta \cos \overline{n-1}\,\beta}{n-1} - \frac{\cos n\beta}{n\beta}\Big) \\ & \quad + e^{-n\xi}\Big(\frac{2 \cos \beta \cos \overline{n-1}\,\beta}{n-1} - \frac{2(1 + \cos 2\beta)\cos n\beta}{n} + \frac{2 \cos \beta \cos (n+1)\beta}{n+1}\Big) \\ & \quad + e^{-(n+2)\xi}\Big(-\frac{\cos n\beta}{n} + \frac{2 \cos \beta \cos(n+1)\beta}{n+1} - \frac{\cos \overline{n+2}\,\beta}{n+2}\Big)\Big\}\cos n\eta\Big]. \end{aligned}$$

The above result has been proved to be true for all $\xi > 0$, in the paper entitled "On the bending of an elliptic plate under certain distribution of load. Part I."

In what follows, we shall use the abbreviation $w(\xi, \eta;\ \beta)$ for the above series.

It is quite easy to show that the above series is absolutely convergent in ξ, η and β for all η and β and for all ξ such that $\xi > 0$. It is uniformly convergent in ξ, η and β for all η and β and all $\xi \geqq \epsilon$ where $\epsilon (> 0)$ is small at pleasure. The series may therefore be multiplied by the bounded continuous function $-c \sin \beta$ and integrated term by term between the limits β_1 and β_2 where $0 \leqq \beta_1 < \beta_2 \leqq \pi$. The sum of the integrated series will represent the integral U given above at all points of the ellipse for which $\xi > 0$ and in particular over $\xi = \alpha$.

We have therefore for $\xi > 0$

$$
\begin{aligned}
U &= -\frac{W}{16\pi D(X_1 - X_2)} \int_{X_2}^{X_1} \{(x-x')^2 + y^2\}[\log\{(x-x')^2 + y^2\} - 2 \log \tfrac{1}{2} c]\, dx' \\
&= -C \int_{\beta_1}^{\beta_2} w_1(\xi, \eta;\ \beta) \sin \beta \, d\beta = -\frac{Wc^2}{32\pi D(\cos \beta_1 - \cos \beta_2)} \Big[\{(2\xi \cosh 2\xi + \xi + 1 + \tfrac{3}{2} e^{-2\xi}) \\
&\qquad \times (\cos \beta_1 - \cos \beta_2) + \tfrac{1}{6}(2 + 2\xi + e^{-2\xi})(\cos 3\beta_1 - \cos 3\beta_2)\} \\
&\quad - \tfrac{1}{4}\{(8\xi \cosh \xi + 2e^{\xi} + 6e^{-\xi} + \tfrac{4}{3} e^{-3\xi})(\cos 2\beta_1 - \cos 2\beta_2) + \tfrac{1}{2}(e^{-\xi} - \tfrac{1}{3} e^{-3\xi})(\cos 4\beta_1 - \cos 4\beta_2)\} \cos \eta \\
&\quad + \{(2\xi + \tfrac{3}{2} + \tfrac{2}{3} e^{-2\xi} + \tfrac{1}{6} e^{-4\xi})(\cos \beta_1 - \cos \beta_2) + \tfrac{1}{6}(1 + e^{-2\xi} - \tfrac{1}{2} e^{-4\xi})(\cos 3\beta_1 - \cos 3\beta_2) \\
&\qquad - \tfrac{1}{30}(e^{-2\xi} - \tfrac{1}{2} e^{-4\xi})(\cos 5\beta_1 - \cos 5\beta_2)\} \cos 2\eta \\
&\quad + \cos 3\eta \Big\{ e^{-\xi} \Big(-\frac{\cos 2\beta_1 - \cos 2\beta_2}{3} + \frac{\cos 4\beta_1 - \cos 4\beta_2}{24} \Big) \\
&\quad + e^{-3\xi} \Big(\frac{\cos 2\beta_1 - \cos 2\beta_2}{24} + \frac{\cos 4\beta_1 - \cos 4\beta_2}{24} - \frac{\cos 6\beta_1 - \cos 6\beta_2}{72} \Big) \\
&\qquad + e^{-5\xi} \Big(\frac{\cos 2\beta_1 - \cos 2\beta_2}{24} - \frac{\cos 4\beta_1 - \cos 4\beta_2}{30} + \frac{\cos 6\beta_1 - \cos 6\beta_2}{120} \Big) \Big\} \\
&\quad + \sum_{n=4}^{\infty} \cos n\eta \Big\{ e^{-(n-2)\xi} \Big(\frac{\cos \overline{n-3}\beta_1 - \cos \overline{n-3}\beta_2}{(n-1)(n-2)(n-3)} - \frac{2(\cos \overline{n-1}\beta_1 - \cos \overline{n-1}\beta_2)}{n(n-1)(n-2)} \\
&\qquad + \frac{\cos \overline{n+1}\beta_1 - \cos \overline{n+1}\beta_2}{(n+1)n(n-1)} \Big) \\
&\quad + e^{-n\xi} \Big(-\frac{\cos \overline{n-3}\beta_1 - \cos \overline{n-3}\beta_2}{n(n-1)(n-3)} + \frac{\cos \overline{n-1}\beta_1 - \cos \overline{n-3}\beta_2}{(n-1)n(n+1)} \\
&\qquad + \frac{\cos \overline{n+1}\beta_1 - \cos \overline{n+1}\beta_2}{(n-1)n(n+1)} - \frac{\cos \overline{n+3}\beta_1 - \cos \overline{n+3}\beta_2}{n(n+1)(n+3)} \Big) \\
&\quad + e^{-(n+2)\xi} \Big\{ \frac{\cos \overline{n-1}\beta_1 - \cos \overline{n-1}\beta_2}{(n-1)n(n+1)} - 2\frac{\cos \overline{n-1}\beta_1 - \cos \overline{n+1}\beta_2}{n(n+1)(n+2)} + \frac{\cos \overline{n+3}\beta_1 - \cos(n+3)\beta_2}{(n+1)(n+2)(n+3)} \Big\} \Big] \qquad (13)
\end{aligned}
$$

The above series is absolutely and uniformly convergent in ξ, η and β for all η and and β and for all $\xi \geqq \epsilon$ where $\epsilon(>0)$ is small at pleasure. For $\xi > 0$, the sum of the above series is U. Now the series obtained by term-by-term differentiation of the above with respect to ξ may likewise be easily proved to be absolutely and uniformly convergent in ξ, η and β for all η and β and for all ξ such that $\xi \geqq \epsilon$, where $\epsilon(>0)$ is small at pleasure. The sum of the differentiated series is therefore equivalent to $\partial U/\partial\xi$ for $\xi > 0$. In particular the value of $\partial U/\partial\xi$ for $\xi = \alpha$ is obtained by putting $\xi = \alpha$ in the derived series.

The boundary conditions $w = 0$ and $\partial w/\partial\xi = 0$ over $\xi = \alpha$ lead to the following formulæ

$$B'_0 \sinh 2\alpha = \frac{Wc^2}{32\pi D(\cos\beta_1 - \cos\beta_2)}\Big[\{\tfrac{3}{2} + (1+\alpha)\sinh 2\alpha - \tfrac{1}{2}\cosh 2\alpha - \cosh^2 2\alpha\}(\cos\beta_1 - \cos\beta_2)$$
$$+ \tfrac{1}{6}\{1 + 2(1+\alpha)\sinh 2\alpha - \cosh 2\alpha\}(\cos 3\beta_1 - \cos 3\beta_2)\Big] \quad (14)$$

$$B_0 \sinh 2\alpha = \frac{Wc^2}{32\pi D(\cos\beta_1 - \cos\beta_2)}\Big[(\tfrac{1}{2} - \tfrac{3}{2}e^{-2\alpha} + \cosh 2\alpha + 2\alpha \sinh 2\alpha)(\cos\beta_1 - \cos\beta_2)$$
$$+ \tfrac{1}{6}(1 - e^{-2\alpha})(\cos 3\beta_1 - \cos 3\beta_2)\Big] \quad (15)$$

$$B'_1(2\sinh 2\alpha + \sinh 4\alpha) = -\frac{Wc^2}{256\pi D(\cos\beta_1 - \cos\beta_2)}\Big[4(\cos 2\beta_1 - \cos 2\beta_2)$$
$$\times\{2 + e^{\alpha}(3\sinh 3\alpha - \cosh 3\alpha) + 3e^{-\alpha}(3\sinh 3\alpha + \cosh 3\alpha)$$
$$+ 4(3\alpha\sinh 3\alpha - \cosh 3\alpha)\cosh\alpha - 4\alpha\sinh\alpha\cosh 3\alpha\}$$
$$-(\cos 4\beta_1 - \cos 4\beta_2)\{1 - e^{-\alpha}(3\sinh 3\alpha + \cosh 3\alpha)\}\Big] \quad (16)$$

$$B_1(2\sinh 2\alpha + \sinh 4\alpha) = \frac{Wc^2}{156\pi D(\cos\beta_1 - \cos\beta_2)}\Big[8(\cos 2\beta_1 - \cos 2\beta_2)$$
$$\times\{1 - 2\cosh^2\alpha + \tfrac{1}{3}e^{-3\alpha}(\sinh\alpha + \cosh 3\alpha\}$$
$$+ \{1 - \tfrac{1}{3}e^{-3\alpha}(3\cosh\alpha + \sinh\alpha)\}(\cos 4\beta_1 - \cos 4\beta_2)\Big] \quad (17)$$

$$B'_2[3\sinh 2\alpha + \sinh 6\alpha] = -4B_0\sinh 4\alpha + \frac{Wc^2}{32\pi D(\cos\beta_1 - \cos\beta_2)}\Big[2\{(4\alpha + 3 + \tfrac{4}{3}e^{-2\alpha} + \tfrac{1}{3}e^{-4\alpha})$$
$$\times\sinh 4\alpha - (1 - \tfrac{2}{3}e^{-2\alpha} - \tfrac{1}{3}e^{-4\alpha})\cosh 4\alpha\}(\cos\beta_1 - \cos\beta_2)$$
$$+ \frac{\cos 3\beta_1 - \cos 3\beta_2}{3}\{(2 + 2e^{-2\alpha} - e^{-4\alpha})\sinh 4\alpha + (e^{-2\alpha} - e^{-4\alpha})\cosh 4\alpha\}$$
$$+ \frac{\cos 5\beta_1 - \cos 5\beta_2}{5}\{(-\tfrac{2}{3}e^{-2\alpha} + \tfrac{1}{3}e^{-4\alpha})\sinh 4\alpha + (-\tfrac{1}{3}e^{-2\alpha} + \tfrac{1}{3}e^{-4\alpha})\cosh 4\alpha\}\Big] \quad (18)$$

$$B_2[3\sinh 2\alpha+\sinh 6\alpha] = 2B_0\sinh 2\alpha-\frac{Wc^2}{32\pi D(\cos\beta_1-\cos\beta_2)}[\{(4\alpha+3+\tfrac{4}{3}e^{-2\alpha}+\tfrac{1}{3}e^{-4\alpha})\sinh 2\alpha$$

$$-(2-\tfrac{4}{3}e^{-2\alpha}-\tfrac{2}{3}e^{-4\alpha})\cosh 2\alpha\}(\cos\beta_1-\cos\beta_2)$$

$$+\frac{\cos 3\beta_1-\cos 3\beta_2}{6}\{(2+2e^{-2\alpha}-e^{-4\alpha})\sinh 2\alpha+2(e^{-2\alpha}-e^{-4\alpha})\cosh 2\alpha\}$$

$$+\frac{\cos 5\beta_1-\cos 5\beta_2}{30}\{(-2e^{-2\alpha}+e^{-4\alpha})\sinh 2\alpha+2(-e^{-2\alpha}+e^{-4\alpha})\cosh 2\alpha\}] \quad (19)$$

$$B'_3[4\sinh 2\alpha+\sinh 8\alpha] = -B_1[3\sinh 4\alpha+2\sinh 6\alpha]+\frac{Wc^2}{16\pi D(\cos\beta_1-\cos\beta_2)}\Big[\frac{\cos 2\beta_1-\cos 2\beta_2}{48}$$

$$\times\{-8e^{-\alpha}(5\sinh 5\alpha+\cosh 5\alpha)+e^{-3\alpha}(5\sinh 5\alpha+3\cosh 5\alpha)+5\}$$

$$+\frac{\cos 4\beta_1-\cos 4\beta_2}{48}\{e^{-\alpha}(5\sinh 5\alpha+\cosh 5\alpha)+e^{-3\alpha}(5\sinh 5\alpha+3\cosh 5\alpha)-4\}$$

$$+\frac{\cos 6\beta_1-\cos 6\beta_2}{144}\{-e^{-3\alpha}(5\sinh 5\alpha+3\cosh 5\alpha)+8\}\Big] \quad (20)$$

$$B_3[4\sinh 2\alpha+\sinh 8\alpha] = B_1[2\sinh 2\alpha+\sinh 4\alpha]-\frac{Wc^2}{16\pi D(\cos\beta_1-\cos\beta_2)}\Big[\frac{\cos 2\beta_1-\cos 2\beta_3}{48}$$

$$\times\{-8e^{-\alpha}(3\sinh 3\alpha+\cosh 3\alpha)+e^{-5\alpha}(3\sinh 3\alpha+5\cosh 3\alpha)+3\}$$

$$+\frac{\cos 4\beta_1-\cos 4\beta_2}{249}\{5e^{-\alpha}(3\sinh 3\alpha+\cosh 3\alpha)-4e^{-5\alpha}(3\sinh 3\alpha+5\cosh 3\alpha)+15\}$$

$$+\frac{\cos 6\beta_1-\cos 6\beta_2}{240}\{e^{-5\alpha}(3\sinh 3\alpha+5\cosh 3\alpha)-5\}\Big] \quad (21)$$

$$B'_n\,[(n+1)\sinh 2\alpha+\sinh(2n+2)\alpha] = -B_{n-2}[n\sinh 4\alpha+2\sinh 2n\alpha]$$

$$+\frac{Wc^2}{32\pi D(\cos\beta_1-\cos\beta_2)}\Big[e^{-(n-2)\alpha}\{(n+2)\sinh(n+2)\alpha+(n-2)\cosh(n+2)\alpha\}$$

$$\times\left\{\frac{\cos(n-3)\beta_1-\cos(n-3)\beta_2}{(n-3)(n-2)(n-1)}-2\frac{\cos(n-1)\beta_1-\cos(n-1)\beta_2}{(n-2)(n-1)n}+\frac{\cos(n+1)\beta_1-\cos(n+1)\beta_2}{(n-1)n(n+1)}\right\}$$

$$+e^{-n\alpha}\{(n+2)\sinh(n+2)\alpha+n\cosh(n+2)\alpha\}\left\{-\frac{\cos(n-3)\beta_1-\cos(n-3)\beta_2}{(n-3(n-1)n}\right.$$

$$\left.+\frac{\cos(n-1)\beta_1-\cos(n-1)\beta_2}{(n-1)n(n+1)}+\frac{\cos(n+1)\beta_1-\cos(n+1)\beta_2}{(n-1)n(n+1)}-\frac{\cos(n+3)\beta_1-\cos(n+3)\beta_2}{n(n+1)(n+3)}\right\}$$

$$+(n+2)\left\{\frac{\cos(n-1)\beta_1-\cos(n-1)\beta_2}{(n-1)n(n+1)}-2\frac{\cos(n+1)\beta_1-\cos(n+1)\beta_2}{n(n+1)(n+2)}+\frac{\cos(n+3)\beta_1-\cos(n+3)\beta_2}{(n+1)(n+2)(n+3)}\right\}\Big] \quad (22)$$

$$B_n[(n+1)\sinh 2\alpha+\sinh(2n+2)\alpha] = B_{n-2}[(n-1)\sinh 2\alpha+\sinh(2n-2)\alpha]-\frac{Wc^2}{32\pi D(\cos\beta_1-\cos\beta_2)}$$

$$\times\Big[e^{-(n-2)\alpha}\{n\sinh n\alpha+(n-2)\cosh n\alpha\}\Big\{\frac{\cos(n-3)\beta_1-\cos(n-3)\beta_2}{(n-3)(n-2)(n-1)}-2\frac{\cos(n-1)\beta_1-\cos(n-1)\beta_2}{(n-2)(n-1)n}$$

$$+\frac{\cos(n+1)\beta_1-\cos(n+1)\beta_2}{(n-1)n(n+1)}\Big\}-\Big\{\frac{\cos(n-3)\beta_1-\cos(n-3)\beta_2}{(n-3)(n-1)}-\frac{\cos(n-1)\beta_1-\cos(n-1)\beta_2}{(n-1)(n+1)}$$

$$-\frac{\cos(n+1)\beta_1-\cos(n+1)\beta_2}{(n-1)(n+1)}+\frac{\cos(n+3)\beta_1-\cos(n+3)\beta_2}{(n+1)(n+3)}\Big\}+e^{-(n+2)\alpha}\{n\sinh n\alpha+(n+2)\cosh n\alpha\}$$

$$\times\Big\{\frac{\cos(n-1)\beta_1-\cos(n-1)\beta_2}{(n-1)n(n+1)}-2\frac{\cos(n+1)\beta_1-\cos(n+1)\beta_2}{n(n+1)(n+2)}+\frac{\cos(n+3)\beta_1-\cos(n+3)\beta_2}{(n+1)(n+2)(n+3)}\Big\}\Big] \qquad (28)$$

We now consider the convergence of the series

$$w' = B'_0+B_0\cosh 2\xi+(B'_1\cosh\xi+B_1\cosh 3\xi)\cos\eta$$

$$+\sum_{n=2}^{\infty}\{B_{n-2}\cosh(n-2)\xi+B'_n\cosh n\xi+B_n\cosh(n+2)\xi\}\cos n\eta$$

We can show, as was done in Part I that for sufficiently large n

$$|B_n| \leqq K/e^{2n\alpha},$$

where K is a positive constant independent of n, and that

$$|B'_n| \leqq K'/e^{2n\alpha},$$

where K' is another positive constant independent of n

We can now prove that each of the series

$$\sum_{n=0}^{\infty} B'_n\cosh n\xi\cos n\eta$$

$$\sum_{n=2}^{\infty} B_{n-2}\cosh(n-2)\xi\cos n\eta$$

$$\sum_{n=0}^{\infty} B_n\cosh(n+2)\xi\cos n\eta$$

is absolutely and uniformly convergent in ξ and η for all η and all ξ such that $|\xi| \leqq 2\alpha-\epsilon$, where $\epsilon(>0)$ is small at pleasure. Each of the series is thus absolutely and uniformly convergent in both the variables throughout the interior as well as the boundary of the ellipse $\xi = 2\alpha-\epsilon$ where $\epsilon(>0)$ is small at pleasure. The same is thus true of the series for w' which is the sum of the above three series.

Following the plan adopted in a previous paper (1948), we can easily demonstrate that the function defined by

$$F_1 = \sum_{n=0}^{\infty} B'_n\cosh n\xi\cos n\eta$$

is the real part of a function of the complex variable z, analytic in the interior of the ellipse $\xi = 2\alpha$. The partial derivatives of F_1 with respect to x and y and also mixed derivatives of F_1 exist and are continuous throughout the interior of the ellipse $\xi = 2\alpha$ The function F_1 is further harmonic in the above range. So it satisfies the equation

$$\nabla_1^4 F_1 = 0$$

throughout the interior of the elliptic plate and also the contributions to the stresses G, H and N due to F_1 are finite and continuous throughout the elliptic plate including the boundary $\xi = \alpha$.

The function defined by the series

$$F_2 = A_0 \cosh 2\xi + A_1 \cosh 3\xi \cos \eta + \sum_{n=2}^{\infty} \{A_{n-2} \cosh(n-2)\xi + A_n \cosh(n+2)\xi\} \cos n\eta$$

is the real part of a complex function of the form

$$(x-iy)(u+iu')$$

where $u+iu'$ is an analytic function of the complex variable z throughout the interior of the ellipse $\xi = 2\alpha$. It is easy to see that F_2 satisfies the differential equation

$$\nabla_1^4 F_2 = 0$$

throughout the elliptic plate and that the partial derivatives of F_2 with respect to x and y and also mixed derivatives exist and are continuous throughout the interior of the ellipse $\xi = 2\alpha$, ensuring thereby the finiteness and continuity of the contributions to the stresses due to F_2 throughout the elliptic plate including the boundary.

We consider with this the fact that we proved elsewhere $\nabla_1^4 U = 0$ throughout the elliptic plate excepting at the points that lie on the line joining the points $(X_2, 0)$ and $(X_1, 0)$ where $-c \leqq X_2 < X_1 \leqq c$. And we also mentioned three results embodied in formulæ (7), (8) and (9) giving the values of $\int N_U ds$ taken along the edges of three rectangles suitably chosen where N_U stands for the contribution to the resultant normal stress N due to U.

Further we have just now seen that

$$\nabla_1^4 w' = \nabla_1^4 (F_1 + F_2) = 0$$

throughout the plate. We may therefore assert that the function defined by

$$w = U + w'$$

where U is given by formula (11) and w' by the formula (12) and the values of the constants appearing in the expression for w' are given by the recurrence formulæ (14), (15), . . ., (23) represent the deflexion of the central plane of a thin elliptic plate clamped along the edge $\xi = \alpha$ under the action of a load of weight W distributed uniformly along the line joining the points, whose cartesian co-ordinates are $(X_2, 0)$ and $(X_1, 0)$ where $-c \leqq X_2 < X_1 \leqq c$ and whose elliptic co-ordinates are $(0, \beta_2)$ and $(0, \beta_1)$ respectively.

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NOTE ON HOMOLOGOUS AND ADIABATIC RADIAL MOTION OF A STAR

By

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Introduction

The radial vibration of gas spheres, under adiabatic conditions, has great importance in the theory of cepheid variables. Investigations of the particular case of the homologous adiabatic pulsation of a homogeneous star has recently been carried out by Rosseland (1943), Bhatnagar and Kothari (1944). The importance of this case lies in the fact that it admits of, as has been shown by later authors, exact solution of the differential equation of vibration which need not necessarily be small. It is, however, worthwhile to examine what conditions are necessary for homologous radial motion of a star under adiabatic conditions. The object of this note is to consider this point and to investigate the general nature of motion (vibratory or otherwise) which can be obtained in the above case.

Equations of Motion

The equations governing the motion are:

$$-\frac{\ddot{r}}{r^2} = 4\pi\frac{\partial P}{\partial m} + \frac{G.m}{r^4} \tag{1}$$

$$4\pi r^2 \rho \frac{\partial r}{\partial m} = 1 \tag{2}$$

$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma \tag{3}$$

Equation (1) is the ordinary equation of radial motion brought to the above form with the help of (2), which is the equation of continuity; and equation (3) represents adiabatic law. m is the mass enclosed within a shell of radius r, and is used here as a Lagrangian co-ordinate. The initial configuration and the initial motion being given, equations (1), (2), (3) determine P, ρ, r as functions of the independent variables m and t.

We now try to obtain the conditions under which homologous radial motion is possible, or, in other words, obtain condition for the existence of a solution separable in m and t as follows:

$$r = r_0(m).f(t) \tag{4}$$

This form was used by Thomas (1930).

CONDITION FOR SEPARABLE SOLUTION

Substituting equation (4) in equation (2), we get

$$\rho = \rho_0/f^3 \tag{5}$$

and (5) in (3), we have

$$P = P_0/f^4 \tag{6}$$

Again substituting from (4), (5), (6) in (1) we get

$$-\frac{f''}{r_0 f^3} = \frac{1}{f^{3\gamma}}.4\pi\frac{dP_0}{dm}+\frac{G.m}{f^4.r_0^4} \tag{7}$$

To proceed further we use a Lemma, previously proved and used by one of us (Bandyopadhyay, 1948). From the Lemma it follows that (7) will be satisfied if either

$$\text{(I)} \quad \frac{f''}{Bf^2} = \frac{A}{f^{3\gamma}} = \frac{1}{f^4} \tag{8}$$

and

$$-\frac{B}{r_0} = \frac{4\pi}{A}.\frac{dP_0}{dm}+\frac{G.m}{r_0^4} \tag{9}$$

or

$$\text{(II)} \quad -\frac{f''}{Cf^2} = \frac{D}{f^{3\gamma}}+\frac{1}{f^4} \tag{10}$$

and

$$\frac{C}{r_0} = \frac{4\pi}{D}.\frac{dP_0}{dm} = \frac{G.m}{r_0^4} \tag{11}$$

A, B, C, D being constants.

Let us consider case I first and suppose that equations (8) and (9) hold. Equation (8) gives

$$\gamma = \frac{4}{3}, \quad A = 1, \quad \frac{f''}{Bf^4} = \frac{1}{f^4} \tag{12}$$

so that aquation (9) reduces to

$$-\frac{B}{r_0} = 4\pi\frac{dP_0}{dm}+\frac{G.m}{r_0^4} \tag{14}$$

The above two equations mean that homologous radial motion is possible for a gas sphere under adiabatic condition if $\gamma = \frac{4}{3}$ and further if the initial configuration satisfies equation (18), B being any constant whatsoever. The form of $f(t)$ in this case is given by the last equation of (12). Besides (18) the given initial configuration need also satisfy the equation of continuity. But two equation, *viz.*, (18), and the equation of continuity are not sufficient to determine the three variables P_0, ρ_c, r_0 uniquely as functions of the independent variable m. Hence any additional equation between the three variables may be adjoined to these two. Physical considerations suggest that a relation between P_0 and ρ_0 may be taken as the most suitable additional equation. Hence we have

Theorem I. *A star with any given initial relation between pressure and density can have homologous radial motion under adiabatic condition if $\gamma=4/3$ and provided certain initial conditions are satisfied. The suitable initial conditions are given by the relation between P_0 and ρ_0, the equation of continuity, and the equation (13).*

Let us next consider case II, *i.e.*, equation (10) and (11). From (11) we have

$$\frac{C}{r_0}=\frac{G.m}{r_0^4}, \quad i.e., \quad m \propto r_0^3 \tag{14}$$

as m and r_0 must be positive it follows $C>0$. Equation (14) means that the star should be initially homogeneous (whatever γ may be). From equation (5) it follows that such a star will remain homogeneous at all times though its density will change with time. The form of $f(t)$ in this case is given by equation (10), which will be discussed in the next section and the initial configuration will be given by the other part of equation (11), *viz.*,

$$\frac{4\pi}{D}\cdot\frac{dP_0}{dm}=\frac{G.m}{r_0^4} \tag{15}$$

Equations (14), (15) and the equation of continuity are sufficient to determine the configuration uniquely. These ensure the usual pressure distribution in a homogeneous star. It may be noted that since

$$\frac{dP_0}{dm}<0$$

it follows from (15) that

$$D<0.$$

Hence we conclude:

Theorem II. *Under adiabatic conditions homogeneous star can have homologous radial motion (no matter what the value of γ may be) and further no other star than a homogeneous one can undergo homologous radial motion if $\gamma \neq 4/3$.*

INVESTIGATION OF THE NATURE OF RADIAL MOTION

We shall now examine the forms of $f(t)$ obtained in the two cases described in the last article to find out which of these can represent vibratory motion and under what conditions.

The equation for Case I is given by

$$f''=B/f^2$$

which on integration gives

$$f'^2=-(2B/f)+E, \tag{16}$$

B and E being constants which can take up any value, positive or negative subject only to the condition of reality of f'. A necessary condition of oscillation is that $f'(t)$ should vanish for two distinct finite values of $f(t)$. This is not possible for equation (16) because the right hand side of (16) can vanish for at most one value of $f(t)$. Hence: *The homologous and adiabatic radial motion of a star for $\gamma=4/3$ can not be oscillatory.*

We next consider equation (10) which gives the differential equation to determine $f(t)$ for Case II. The considerations of previous article lead to the restrictions

$$C > 0, \qquad D < 0$$

Putting

$$f(t) = \frac{\xi(t)}{(-D)^{\frac{1}{4-3\gamma}}}, \qquad p^2 = \frac{C}{(-D)^{\frac{2}{4-3\gamma}}} \tag{17}$$

We have

$$\xi'' = p^2\left(-\frac{1}{\xi^2} + \frac{1}{\xi^{3\gamma-2}}\right) \tag{18}$$

on integration we get

$$\xi'^2 = p^2\left(-F + \frac{2}{\xi} - \frac{2}{3(\gamma-1)}\cdot\frac{1}{\xi^{3\gamma-3}}\right) \tag{19}$$

F being a constant. Putting $\xi=1$ in equation (18) we find $\xi'=0$. We shall call $\xi=1$ the equilibrium configuration of the star. In vibration, therefore ξ' will become zero once for a value of ξ greater than unity and once again for a value of ξ less than unity.

We enumerate the conditions of vibration thus:

(*i*) ξ' and therefore the right hand side of the equation (19), should vanish for two values of ξ, *viz.*, for $\xi_1 > 1$ and $\xi_2 < 1$.

(*ii*) The value of ξ'' must be negative for $\xi=\xi_1$ and positive for $\xi=\xi_2$.

(*iii*) The time taken in passing from ξ_1 to ξ_2 should be finite. This will be ensured provided the roots, ξ_1 and ξ_2 (of the right hand side of (19) equated to zero) are not repeated roots.

We shall examine condition (*ii*) first. When $\gamma < \frac{4}{3}$, then $3\gamma-2 < 2$, so that for $\xi > 1$

$$\xi^{3\gamma-2} < \xi^2$$

that is

$$\frac{1}{\xi^{3\gamma-2}} - \frac{1}{\xi^2} > 0,$$

So that ξ'' is positive for $\xi > 1$. Thus condition (*ii*) can not be satisfied for $\gamma < 4/3$. It can similarly be shown that condition (*ii*) is satisfied for $\gamma > 4/3$.

We shall now examine condition (*i*) confining ourselves to the case $\gamma > 4/3$ only. We write

$$\frac{1}{\xi^{3\gamma-3}} = \frac{1}{\xi^{p/q}} = x^p, \text{ say}$$

where p, q are positive integers; γ being assumed rational. $\gamma > 4/3$ now leads to the condition $p > q$; the right hand side of (19) equated to zero takes the form

$$\varphi(x) \equiv -F + 2x^q - \frac{2q}{p}x^p = 0, \quad (p > q) \tag{20}$$

By Descartes' rule of signs this can not have more positive roots than the changes of sign. From this we should obtain (after examining the function φ)

$$F > 0,$$

as necessary conditions for $\varphi(x)$ to have two positive roots. We also note:

$$\varphi(0) = -F$$

$$\varphi(1) = -F+2\left(1-\frac{1}{3\gamma-3}\right)$$

$$\varphi(\infty) = -\infty, \text{ (negative)}$$

so that $\varphi(x)$ will have two positive roots, one less and the other greater than unity (just like ξ) if $\varphi(1) > 0$. That this last condition must hold can be seen from the requirement $\xi'^2 > 0$. for $\xi = 1$. Thus condition (*ii*) is fully satisfied for the case $\gamma > 4/3$ provided

$$0 < F < 2\left(1-\frac{1}{3\gamma-3}\right)$$

We now take up condition (*iii*). Differentiating the left hand side of (20) we get

$$\varphi'(x) = 2q(x^{q-1}-x^{p-1})$$

which shows that besides $x = 0$, $x = 1$ is the only positive root of $\varphi'(x)$. But the two positive roots of $\varphi(x)$ are different from zero and unity. So $\varphi'(x)$ and $\varphi(x)$ has no common positive roots. Condition (*iii*) is thus fulfilled. We thus conclude that *the adiabatic homologous motion of a gas sphere can have the character of vibration if*

$$\gamma > \tfrac{4}{3}, \qquad 0 < F < 2\left(1-\frac{1}{3\gamma-3}\right).$$

We have seen before that vibration takes place only through homogeneous configurations. The equation (19) has been solved exactly for $\gamma = 5/3$, $3/2$ $13/9$ (Bhatnagar, 1946), The present discussion establishes the possibility of adiabatic oscillation of a homogeneous gas sphere through homologous pulsation configurations for all $\gamma > 4/3$. Moreover such homologous oscillations are possible only for homogeneous spheres. It may be pointed out that the inequality

$$F > 0$$

signifies

$$\xi'^2_{\max} < \frac{2}{3}\, G\rho_0 . \frac{3\gamma-4}{\gamma-1}$$

ρ_0 being equilibrium density. F is thus connected with the energy of the configuration.

In conclusion we take the privilege of acknowledging our grateful thanks to Prof. N. R. Sen for his kind interest and helpful suggestions.

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ON A CONTINUED FRACTION OF A GENERAL TYPE

By

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Milne-Thomson (1933 *a*, *b*) first exhibited a simple continued fraction in the form of a continued matrix product. Upadhyay and Das Gupta (1947) discussed continued fractions of a general type where the various partial quotients were themselves continued fractions. The author in this paper discusses some of the formulæ obtained in the above paper, and extends the idea of continuing the type referred to by further assuming the partial quotients of the general type as being expressible themselves by continued fractions.

Notation

Let $a\begin{pmatrix} n \\ m \end{pmatrix}$ denote the continued fraction, such that

$$a\begin{pmatrix} n \\ m \end{pmatrix} = a_1(m) + \frac{1}{a_2(m)+}\,\frac{1}{a_3(m)+}\cdots\frac{1}{a_n(m)}$$

where

$$a_i(m) = a_{i1} + \frac{1}{a_{i2}+}\,\frac{1}{a_{i3}+}\cdots\frac{1}{a_{im}}.$$

Let $a\begin{pmatrix} n \\ m \\ l \end{pmatrix}$ denote the continued fraction, such that

$$a\begin{pmatrix} n \\ m \\ l \end{pmatrix} = a_1\begin{pmatrix} m \\ l \end{pmatrix} + \frac{1}{a_2\begin{pmatrix} m \\ l \end{pmatrix}+}\,\frac{1}{a_3\begin{pmatrix} m \\ l \end{pmatrix}+}\cdots\frac{1}{a_n\begin{pmatrix} m \\ l \end{pmatrix}}$$

where

$$a_i\begin{pmatrix} m \\ l \end{pmatrix} = a_{i1}(l) + \frac{1}{a_{i2}(l)+}\,\frac{1}{a_{i3}(l)+}\cdots\frac{1}{a_{im}(l)},$$

where again

$$a_{ij}(l) = a_{ij1} + \frac{1}{a_{ij2}+}\,\frac{1}{a_{ij3}+}\cdots\frac{1}{a_{ijl}}.$$

Let $a\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ r & s & t & \cdots & v \end{pmatrix}$ denote the continued fraction, such that

$$a\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ r & s & t & \cdots & v \end{pmatrix} = a_1(r) + \frac{1}{a_2(s)+}\,\frac{1}{a_3(t)+}\cdots\frac{1}{a_n(v)}.$$

Further denote the numerators of the convergents

$$a\binom{n}{m},\quad a_i(m),\quad a\begin{pmatrix} n\\ m\\ l\end{pmatrix},\quad a\begin{pmatrix}1 & 2 & 3 & \cdots & n\\ r & s & t & \cdots & v\end{pmatrix}$$

by

$$p\binom{n}{m},\quad p_i(m),\quad p\begin{pmatrix} n\\ m\\ l\end{pmatrix},\quad p\begin{pmatrix}1 & 2 & 3 & \cdots & n\\ r & s & t & \cdots & v\end{pmatrix}$$

and the corresponding denominators by

$$q\binom{n}{m},\quad q_i(m),\quad q\begin{pmatrix} n\\ m\\ l\end{pmatrix};\quad q\begin{pmatrix}1 & 2 & 3 & \cdots & n\\ r & s & t & \cdots & v\end{pmatrix}.$$

(A)

1. Consider the continued fraction of the form

$$A_n = \frac{\alpha_1}{\beta_1} + \cfrac{1}{\cfrac{\alpha_2}{\beta_2} + \cdots \cfrac{1}{\cfrac{\alpha_n}{\beta_n}}}$$

where if $A_n = p_n/q_n$, we can, by induction, prove,

$$p_n = \alpha_n p_{n-1} + \beta_{n-1}\beta_n p_{n-2},$$

$$q_n = \alpha_n q_{n-1} + \beta_{n-1}\beta_n q_{n-2}.$$

So,

$$\begin{bmatrix} p_n\\ q_n\end{bmatrix} = \begin{bmatrix} p_{n-1} & p_{n-2}\\ q_{n-1} & q_{n-2}\end{bmatrix}\begin{bmatrix} \alpha_n\\ \beta_{n-1}\beta_n\end{bmatrix} = \begin{bmatrix} p_{n-2} & p_{n-3}\\ q_{n-2} & q_{n-3}\end{bmatrix}\begin{bmatrix} \alpha_{n-1} & 1\\ \beta_{n-2}\beta_{n-1} & 0\end{bmatrix}\begin{bmatrix} \alpha_n\\ \beta_{n-1}\beta_n\end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 & 1\\ \beta_1 & 0\end{bmatrix}\begin{bmatrix} \alpha_2 & 1\\ \beta_1\beta_2 & 0\end{bmatrix}\cdots\begin{bmatrix} \alpha_{n-1} & 1\\ \beta_{n-2}\beta_{n-1} & 0\end{bmatrix}\begin{bmatrix} \alpha_n\\ \beta_{n-1}\beta_n\end{bmatrix}.$$

That is,

$$\begin{bmatrix} p_n\\ q_n\end{bmatrix} = \begin{bmatrix} \alpha_1 & 1\\ \beta_1 & 0\end{bmatrix}\begin{bmatrix} 1 & 0\\ 0 & \beta_1\end{bmatrix}\begin{bmatrix} \alpha_2 & 1\\ \beta_2 & 0\end{bmatrix}\begin{bmatrix} 1 & 0\\ 0 & \beta_2\end{bmatrix}\cdots\begin{bmatrix} \alpha_{n-1} & 1\\ \beta_{n-1} & 0\end{bmatrix}\begin{bmatrix} 1 & 0\\ 0 & \beta_{n-1}\end{bmatrix}\begin{bmatrix} \alpha_n\\ \beta_n\end{bmatrix}. \tag{1}$$

Also,

$$\begin{bmatrix} p_n\\ q_n\end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1\\ \beta_1 & 0\end{bmatrix}\begin{bmatrix} \alpha_2 & \beta_2\\ \beta_2 & 0\end{bmatrix}\cdots\begin{bmatrix} \alpha_{n-1} & \beta_{n-1}\\ \beta_{n-1} & 0\end{bmatrix}\begin{bmatrix} \alpha_n\\ \beta_n\end{bmatrix}. \tag{2}$$

Hence for the continued fraction $a\begin{pmatrix}1 & 2 & \cdots & \overline{n-1} & n\\ r & s & \cdots & u & v\end{pmatrix}$, it follows from (1) that

$$\begin{bmatrix} p\begin{pmatrix}1&2&\cdots&\overline{n-1}&n\\ r&s&\cdots&u&v\end{pmatrix}\\ q\begin{pmatrix}1&2&\cdots&\overline{n-1}&n\\ r&s&\cdots&u&v\end{pmatrix}\end{bmatrix}=\begin{bmatrix}p_1(r)&1\\q_1(r)&0\end{bmatrix}\begin{bmatrix}1&0\\0&q_1(r)\end{bmatrix}\begin{bmatrix}p_2(s)&1\\q_2(s)&0\end{bmatrix}\begin{bmatrix}1&0\\0&q_2(s)\end{bmatrix}\cdots\begin{bmatrix}p_{n-1}(u)&1\\q_{n-1}(u)&0\end{bmatrix}\begin{bmatrix}1&0\\0&q_{n-1}(u)\end{bmatrix}\begin{bmatrix}p_n(v)\\q_n(v)\end{bmatrix}. \qquad (3)$$

and from (2) that

$$\begin{bmatrix} p\begin{pmatrix}1&2&\cdots&\overline{n-1}&n\\ r&s&\cdots&u&v\end{pmatrix}\\ q\begin{pmatrix}1&2&\cdots&\overline{n-1}&n\\ r&s&\cdots&u&v\end{pmatrix}\end{bmatrix}=\begin{bmatrix}p_1(r)&q_1(r)\\q_1(r)&0\end{bmatrix}\begin{bmatrix}p_2(s)&q_2(s)\\q_2(s)&0\end{bmatrix}\cdots\begin{bmatrix}p_{n-1}(u)&q_{n-1}(u)\\q_{n-1}(u)&0\end{bmatrix}\begin{bmatrix}p_n(v)\\q_n(v)\end{bmatrix}. \qquad (4)$$

2. Now in the continued fraction $a_i(j)$, we have

$$p_i(j) = a_{ij}p_i(j-1)+p_i(j-2),$$

$$q_i(j) = a_{ij}q_i(j-1)+q_i(j-2).$$

So,

$$\begin{bmatrix}p_i(j)&1\\q_i(j)&0\end{bmatrix}=\begin{bmatrix}p_i(j-1)&p_i(j-2)&1\\q_i(j-1)&q_i(j-2)&0\end{bmatrix}\begin{bmatrix}a_{ij}&0\\1&0\\0&1\end{bmatrix}=\begin{bmatrix}p_i(j-2)&p_i(j-3)&1\\q_i(j-2)&q_i(j-3)&0\end{bmatrix}\begin{bmatrix}a_{i\overline{j-1}}&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}a_{ij}&0\\1&0\\0&1\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}a_{i1}&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}a_{i2}&1&0\\1&0&0\\0&0&1\end{bmatrix}\cdots\begin{bmatrix}a_{i\overline{j-1}}&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}a_{ij}&0\\1&0\\0&1\end{bmatrix}. \qquad (5)$$

And

$$\begin{bmatrix}1&0\\0&q_i(j)\end{bmatrix}=\begin{bmatrix}1&0&0\\0&q_i(j)&q_i(j-1)\end{bmatrix}\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\0&q_i(j-1)&q_i(j-2)\end{bmatrix}\begin{bmatrix}1&0&0\\0&a_{ij}&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&a_{i2}&1\\0&1&0\end{bmatrix}\cdots\begin{bmatrix}1&0&0\\0&a_{ij}&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}. \qquad (6)$$

Also,

$$\begin{bmatrix}p_n(v)\\q_n(v)\end{bmatrix}=\begin{bmatrix}a_{n1}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{n2}&1\\1&0\end{bmatrix}\cdots\begin{bmatrix}a_{n\overline{v-1}}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{nv}\\1\end{bmatrix}, \qquad (7)$$

Hence by the use of (5), (6) and (7), we have from (3) the result:

$$\begin{bmatrix} p\begin{pmatrix} 1 & 2 & \cdots & \overline{n-1} & n \\ r & s & \cdots & u & v \end{pmatrix} \\ q\begin{pmatrix} 1 & 2 & \cdots & \overline{n-1} & n \\ r & s & \cdots & u & v \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{12} & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} a_{1\overline{r-1}} & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{1r} & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{12} & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{1\overline{r-1}} & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{1r} & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{21} & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{22} & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} a_{2\overline{s-1}} & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{2s} & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{22} & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{2\overline{s-1}} & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{2s} & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\times \quad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$\times \quad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$\times \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{\overline{n-1}1} & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{\overline{n-1}2} & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} a_{\overline{n-1}\,\overline{u-1}} & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{\overline{n-1}u} & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{\overline{n-1}2} & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{\overline{n-1}\,\overline{u-1}} & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{\overline{n-1}u} & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} a_{n1} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n2} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{n\overline{v-1}} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{nv} \\ 1 \end{bmatrix} \tag{8}$$

3. We now introduce the arrow operator $(\overrightarrow{\quad})$ which is defined by the relation

$$\overrightarrow{\begin{bmatrix} N \\ D \end{bmatrix}} = \begin{bmatrix} N & D \\ D & O \end{bmatrix}.$$

The operator acts upon the 2 × 1 matrix $\begin{bmatrix} N \\ D \end{bmatrix}$ and converts it into a 2 × 2 matrix $\begin{bmatrix} N & D \\ D & O \end{bmatrix}$

Then from the definition of the operator, we have

$$\overrightarrow{\begin{bmatrix} p_i(j) \\ q_i(j) \end{bmatrix}} = \begin{bmatrix} p_i(j) & q_i(j) \\ q_i(j) & 0 \end{bmatrix}. \tag{9}$$

So, by using the operational notation as in (9), it follows from (4) that

$$\begin{bmatrix} p\begin{pmatrix}1 & 2 & \cdots & \overline{n-1} & n\\ r & s & \cdots & u & v\end{pmatrix}\\ \\ q\begin{pmatrix}1 & 2 & \cdots & \overline{n-1} & n\\ r & s & \cdots & u & v\end{pmatrix}\end{bmatrix} = \overrightarrow{\begin{bmatrix}p_1(r)\\ \\ q_1(r)\end{bmatrix}}\overrightarrow{\begin{bmatrix}p_2(s)\\ \\ q_2(s)\end{bmatrix}}\cdots\overrightarrow{\begin{bmatrix}p_{n-1}(u)\\ \\ q_{n-1}(u)\end{bmatrix}}\begin{bmatrix}p_n(v)\\ \\ q_n(v)\end{bmatrix}, \tag{10}$$

Here again

$$\begin{bmatrix}p_i(j)\\ \\ q_i(j)\end{bmatrix} = \begin{bmatrix}a_{i1} & 1\\ \\ 1 & 0\end{bmatrix}\begin{bmatrix}a_{i2} & 1\\ \\ 1 & 0\end{bmatrix}\cdots\begin{bmatrix}a_{i\overline{j-1}} & 1\\ \\ 1 & 0\end{bmatrix}\begin{bmatrix}a_{ij}\\ \\ 1\end{bmatrix}$$

Hence the result (10) can be exhibited as a formula:

$$\begin{aligned}\begin{bmatrix} p\begin{pmatrix}1 & 2 & \cdots & \overline{n-1} & n\\ r & s & \cdots & u & v\end{pmatrix}\\ \\ q\begin{pmatrix}1 & 2 & \cdots & \overline{n-1} & n\\ r & s & \cdots & u & v\end{pmatrix}\end{bmatrix} &= \overrightarrow{\begin{bmatrix}a_{11} & 1\\ \\ 1 & 0\end{bmatrix}\begin{bmatrix}a_{12} & 1\\ \\ 1 & 0\end{bmatrix}\cdots\begin{bmatrix}a_{1\overline{r-1}} & 1\\ \\ 1 & 0\end{bmatrix}\begin{bmatrix}a_{1r}\\ \\ 1\end{bmatrix}}\\ &\times\overrightarrow{\begin{bmatrix}a_{31} & 1\\ \\ 1 & 0\end{bmatrix}\begin{bmatrix}a_{22} & 1\\ \\ 1 & 0\end{bmatrix}\cdots\begin{bmatrix}a_{2\overline{s-1}} & 1\\ \\ 1 & 0\end{bmatrix}\begin{bmatrix}a_{2s}\\ \\ 1\end{bmatrix}}\times\;\cdots\\ &\times\overrightarrow{\begin{bmatrix}a_{\overline{n-1}1} & 1\\ \\ 1 & 0\end{bmatrix}\begin{bmatrix}a_{\overline{n-2}1} & 1\\ \\ 1 & 0\end{bmatrix}\cdots\begin{bmatrix}a_{\overline{n-1}\,\overline{u-1}} & 1\\ \\ 1 & 0\end{bmatrix}\begin{bmatrix}a_{\overline{n-1}u}\\ \\ 1\end{bmatrix}}\\ &\times\begin{bmatrix}a_{n1} & 1\\ \\ 1 & 0\end{bmatrix}\begin{bmatrix}a_{n2} & 1\\ \\ 2 & 0\end{bmatrix}\cdots\begin{bmatrix}a_{n\,v-1} & 1\\ \\ 1 & 0\end{bmatrix}\begin{bmatrix}a_{nv}\\ \\ 1\end{bmatrix}.\end{aligned} \tag{11}$$

This result (11) is perfectly general and will replace the results (14), (15), (16), (17), (18), (19) and (20) of the paper by Upadhyay and Das-Gupta (1947).

(B)

4. In the case, when $r=s=\cdots=u=v(=\boldsymbol{m})$, the result (11) can be expressed as

$$\begin{bmatrix} p\begin{pmatrix}1 & 2 & \cdots & \overline{n-1} & n\\ m & m & \cdots & m & m\end{pmatrix}\\ \\ q\begin{pmatrix}1 & 2 & \cdots & \overline{n-1} & n\\ m & m & \cdots & m & m\end{pmatrix}\end{bmatrix} = \left\{\overrightarrow{\prod_{i=1}^{n-1}\left(\prod_{j=1}^{m-1}\overrightarrow{\begin{bmatrix}a_{ij}\\ 1\end{bmatrix}}\right)\begin{bmatrix}a_{im}\\ 1\end{bmatrix}}\right\}\left(\prod_{j=1}^{m-1}\overrightarrow{\begin{bmatrix}a_{nj}\\ 1\end{bmatrix}}\right)\begin{bmatrix}a_{nm}\\ 1\end{bmatrix} \tag{12}$$

Since, from the notations, we have

$$\begin{bmatrix} p\begin{pmatrix}1 & 2 & \cdots & n\\ m & m & \cdots & m\end{pmatrix}\\ \\ q\begin{pmatrix}1 & 2 & \cdots & n\\ m & m & \cdots & m\end{pmatrix}\end{bmatrix} = \begin{bmatrix} p\begin{pmatrix}n\\ m\end{pmatrix}\\ \\ q\begin{pmatrix}n\\ m\end{pmatrix}\end{bmatrix}.$$

the result (12) can briefly be stated as a formula:

$$\begin{bmatrix} p\begin{pmatrix} n \\ m \end{pmatrix} \\ q\begin{pmatrix} n \\ m \end{pmatrix} \end{bmatrix} = \left\{ \overrightarrow{\prod_{i=1}^{n-1} \left(\overrightarrow{\prod_{j=1}^{m-1}} \begin{bmatrix} a_{ij} \\ 1 \end{bmatrix} \right) \begin{bmatrix} a_{im} \\ 1 \end{bmatrix}} \right\} \left(\overrightarrow{\prod_{j=1}^{m-1}} \begin{bmatrix} a_{nj} \\ 1 \end{bmatrix} \right) \begin{bmatrix} a_{nm} \\ 1 \end{bmatrix}. \tag{13}$$

5. An extension of the above result·

In the simple continued fraction $a_i(m)$, we know

$$\begin{bmatrix} p_i(m) \\ q_i(m) \end{bmatrix} = \left(\overrightarrow{\prod_{j=1}^{m-1}} \begin{bmatrix} a_{ij} \\ 1 \end{bmatrix} \right) \begin{bmatrix} a_{im} \\ 1 \end{bmatrix}.$$

By substituting this value in (13) we get

$$\begin{bmatrix} p\begin{pmatrix} n \\ m \end{pmatrix} \\ q\begin{pmatrix} n \\ m \end{pmatrix} \end{bmatrix} = \left(\overrightarrow{\prod_{i=1}^{n-1}} \begin{bmatrix} p_i(m) \\ q_i(m) \end{bmatrix} \right) \begin{bmatrix} p_n(m) \\ q_n(m) \end{bmatrix}. \tag{14}$$

Extending this result to the continued fraction $a\begin{pmatrix} n \\ m \\ l \end{pmatrix}$, we have

$$\begin{bmatrix} p\begin{pmatrix} n \\ m \\ l \end{pmatrix} \\ q\begin{pmatrix} n \\ m \\ l \end{pmatrix} \end{bmatrix} = \left(\overrightarrow{\prod_{i=1}^{n-1} \begin{bmatrix} p_i\begin{pmatrix} m \\ l \end{pmatrix} \\ q_i\begin{pmatrix} m \\ l \end{pmatrix} \end{bmatrix}} \right) \begin{bmatrix} p_n\begin{pmatrix} m \\ l \end{pmatrix} \\ q_n\begin{pmatrix} m \\ l \end{pmatrix} \end{bmatrix}. \tag{15}$$

Here, it follows from (13) that

$$\begin{bmatrix} p_i\begin{pmatrix} m \\ l \end{pmatrix} \\ q_i\begin{pmatrix} m \\ l \end{pmatrix} \end{bmatrix} = \left\{ \overrightarrow{\prod_{j=1}^{m-1} \left(\overrightarrow{\prod_{k=1}^{l-1}} \begin{bmatrix} a_{ijk} \\ 1 \end{bmatrix} \right) \begin{bmatrix} a_{ijl} \\ 1 \end{bmatrix}} \right\} \left(\overrightarrow{\prod_{k=1}^{l-1}} \begin{bmatrix} a_{imk} \\ 1 \end{bmatrix} \right) \begin{bmatrix} a_{iml} \\ 1 \end{bmatrix}$$

Substituting this value in (15) we get the formula:

$$\begin{aligned} \begin{bmatrix} p\begin{pmatrix} n \\ m \\ l \end{pmatrix} \\ q\begin{pmatrix} n \\ m \\ l \end{pmatrix} \end{bmatrix} &= \left(\overrightarrow{\prod_{i=1}^{n-1} \left\{ \overrightarrow{\prod_{j=1}^{m-1} \left(\overrightarrow{\prod_{k=1}^{l-1}} \begin{bmatrix} a_{ijk} \\ 1 \end{bmatrix} \right) \begin{bmatrix} a_{ijl} \\ 1 \end{bmatrix}} \right\} \left(\overrightarrow{\prod_{k=1}^{l-1}} \begin{bmatrix} a_{imk} \\ 1 \end{bmatrix} \right) \begin{bmatrix} a_{iml} \\ 1 \end{bmatrix}} \right) \\ &\quad \times \left\{ \overrightarrow{\prod_{j=1}^{m-1} \left(\overrightarrow{\prod_{k=1}^{l-1}} \begin{bmatrix} a_{njk} \\ 1 \end{bmatrix} \right) \begin{bmatrix} a_{njl} \\ 1 \end{bmatrix}} \right\} \left(\overrightarrow{\prod_{k=1}^{l-1}} \begin{bmatrix} a_{nmk} \\ 1 \end{bmatrix} \right) \begin{bmatrix} a_{iml} \\ 1 \end{bmatrix} \end{aligned} \tag{16}$$

The method is perfectly general and the formula can be extended to any number of elements.

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ON THE CUMULANTS OF β_8

By

D. P. Banerjee

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Hsu (1939) has found the moments of b_2. Here I find the cumulants of $\beta_3 = \mu_8/\mu_2^4$.

Now (Kendall, 1945, p. 271)

$$\mu_8 = k_8 + 28k_6k_2 + 56k_5k_3 + 35k_4^2 + 260k_4k_2^2 + 280k_3^2k_2 + 105k_2^4.$$

Hence

$$\mu(\beta_3) = \mu(82^{-4}) + 28\mu(62^{-3}) + 56\mu(5.3\,2^{-4}) + 35\mu(4^22^{-4}) + 210\mu(42^{-2}) + 280\mu(3^22^{-3}) + 105n.$$

Again (Kendall, 1945, p. 289)

$$\mu(\cdots 5^c4^b3^a) = \mu(\cdots 5^c4^b3^a2^{-j}).k_2^j.\frac{(n-1)(n+1)\cdots(n+2j-3)}{(n-1)^j},$$

Now

$$\mu(8) = k_8, \quad \mu(6) = k_6, \quad \mu(4) = k_1, \quad \mu(5.3) = k(53)$$

$$\mu(4^2) = k(4^2) + k_1^2, \quad \mu(3^2) = k(3^2) + k^2(3) = k(3^2) + k_3^2.$$

Neglecting $k_3, k_4, \cdots, k_8$ we have

$$k(4^2) = \frac{24n(n+1)}{(n-1)(n-2)(n-3)}k_2^4, \qquad \mu(8) = 0 = \mu(6) = \mu(4) = \mu(5.3),$$

$$k(3^2) = \frac{6n\,k_2^3}{(n-1)(n-2)}, \qquad \mu(4^2) = \frac{24n(n+1)}{(n-1)(n-2)(n-3)}k_2^4,$$

$$k(53) = 0, \qquad \mu(3^2) \doteq \frac{6n\,k_2^3}{(n-1)(n-2)}.$$

We have

$$\mu(\beta_3) = 35\,\frac{24n(n+1)(n-1)(n+1)\cdots(n+5)}{(n-1)(n-2)(n-3)(n-1)^4}k_2^8 + 280\frac{(n-1)(n+1)(n+3)}{(n-1)^3}.\frac{6n}{(n-1)(n-2)}k_2^6 + 105n.$$

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ON THE FLEXURE OF A BEAM WHOSE CROSS-SECTION IS BOUNDED PARTLY BY A STRAIGHT LINE

By
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1. It has recently been shewn (Ghosh, 1947a) that the problem of flexure of an isotropic elastic beam can be solved by a conformal representation of the area of the cross-section of the beam on the unit circle. The function which makes this conformal representation is often of a very complicated form, and we encounter considerable difficulty in evaluating the integrals appearing in the solution. But when the cross-section is bounded partly by a straight line, and forms half of an area which is symmetrical about this straight line and which can be represented conformally on the unit circle by a function of comparatively simple form, we can use, with advantage, the method developed in a previous paper (Ghosh, 1947b) for solving the problem of torsion of a beam of the same cross-section.

2. Let C_1 be a closed curve in the z-plane ($z = x+iy$) symmetrical about the real axis, and let the cross-section R of the beam be the part of the area within C_1 above the real axis, so that the boundary C of R consists of the upper half of C_1, and the part of the real axis lying within C_1. Let $x = a$, $y = b$ give the centroid of R. Then $x' = x-a$, $y' = y-b$ are the coordinates of the point (x, y) of the cross-section, referred to parallel axes through the centroid.

The solution of the flexure problem consists in determining two plane harmonic functions ψ_1, ψ_2 in R, which satisfy respectively the conditions (Ghosh, 1947a, p. 4)

$$\psi_1 = \int_A^P [(1+\sigma)x'^2 - \sigma y'^2]\,dy' + \text{const.}, \tag{2.1}$$

$$\psi_2 = -\int_A^P [(1+\sigma)y'^2 - \sigma x'^2]\,dx' + \text{const.}, \tag{2.2}$$

on the boundary C. Substituting $x' = x-a$, $y' = y-b$, we find, after partial integration and omission of additive constants, that the boundary values of ψ_1, ψ_2 are respectively

$$\psi_1 = [(1+\sigma)a^2 - \sigma b^2]y - (1+\sigma)axy + \sigma by^2 - \tfrac{1}{3}(1+2\sigma)y^3 + (1+\sigma)\boldsymbol{I}\int_A^P (z-a)\bar{z}\,dz, \tag{2.3}$$

$$\begin{aligned}\psi_2 = {} & [\sigma a^2 - (1+\sigma)b^2]x - \sigma ax^2 + (1-\sigma)bxy + \tfrac{1}{3}\sigma x^3 \\ & - (1+\sigma)b\boldsymbol{I}\int_A^P \bar{z}\,dz + \tfrac{1}{4}(1+\sigma)\boldsymbol{R}\int_A^P (z-\bar{z})^2\,dz,\end{aligned} \tag{2.4}$$

where $\boldsymbol{R}$ and $\boldsymbol{I}$ placed before an expression denote respectively the real and the imaginary parts of that expression.

If we put

$$\Psi_1 = \psi_1 - [(1+\sigma)a^2 - \sigma b^2]y + (1+\sigma)axy, \tag{2.5}$$

$$\Psi_2 = \psi_2 - [\sigma a^2 - (1+\sigma)b^2]x + \sigma a(x^2 - y^2) - (1+\sigma)bxy - \tfrac{1}{3}\sigma(x^3 - 3xy^2), \tag{2.6}$$

we see that Ψ_1, Ψ_2 are plane harmonic functions in R, and satisfy the conditions

$$\Psi_1 = \sigma b y^2 - \tfrac{1}{3}(1+2\sigma)y^3 + (1+\sigma)\boldsymbol{I}\int_A^P (z-a)\bar{z}\,dz, \tag{2.7}$$

$$\Psi_2 = -\sigma a y^2 + \sigma x y^2 - (1+\sigma)b\boldsymbol{I}\int_A^P \bar{z}\,dz + \tfrac{1}{4}(1+\sigma)\boldsymbol{R}\int_A^P (z-\bar{z})^2\,dz, \tag{2.8}$$

respectively on the boundary C of R.

Taking A on the real axis, we see that $\Psi_1 = \Psi_2 = 0$ on the real axis. If we therefore put

$$F_1 = \Phi_1 + i\Psi_1, \qquad F_2 = \Phi_2 + i\Psi_2,$$

we can, by choosing Φ_1, Φ_2 suitably, make F_1, F_2 analytic in R, and real on the real axis, so that by Schwarz's principle of reflection, they can be continued analytically to the part of the area within C_1 below the real axis, Ψ_1, Ψ_2 taking the values

$$\Psi_1 = -\sigma b y^2 - \tfrac{1}{3}(1+2\sigma)y^3 + (1+\sigma)\boldsymbol{I}\int_A^P (z-a)\bar{z}\,dz, \tag{2.9}$$

$$\Psi_2 = \sigma a y^2 - \sigma x y^2 - (1+\sigma)b\boldsymbol{I}\int_A^P \bar{z}\,dz - \tfrac{1}{4}(1+\sigma)\boldsymbol{R}\int_A^P (z-\bar{z})^2\,dz, \tag{2.10}$$

respectively on the part of C_1 below the real axis.

3. Let us now suppose that

$$z = \omega(\zeta)$$

maps the area within C_1 in the z-plane conformally on the unit circle in the ζ-plane. Then it maps the domain R in the z-plane conformally on the upper half of the unit circle, the straight boundary of R on the real axis corresponding to the diameter of the unit circle on the real axis.

The functions $F_1(\zeta)$ and $F_2(\zeta)$ are analytic within the unit circle, and their imaginary parts Ψ_1 and Ψ_2 take up values given by (2.7) and (2.8) respectively on the upper half α of the circumference γ of the unit circle, and the values given by (2.9) and (2.10) respectively on the lower half β of the circumference γ. With the help of Schwarz's formula in its modified form used in a previous paper (Ghosh, 1947a, p. 2) we get

$$F_1(\zeta) = -\frac{\sigma b}{4\pi}\left[\int_\alpha \frac{[\omega(t)-\bar{\omega}(1/t)]^2}{t-\zeta}\,dt - \int_\beta\right] + \frac{1+2\sigma}{24\pi i}\int_\gamma \frac{[\omega(t)-\bar{\omega}(1/t)]^3}{t-\zeta}\,dt + \frac{1+\sigma}{2\pi i}\int_\gamma \frac{F(t)-\bar{F}(1/t)}{t-\zeta}\,dt, \tag{3.1}$$

$$F_2(\zeta) = \frac{\sigma a}{4\pi}\left[\int_\alpha \frac{[\omega(t)-\bar{\omega}(1/t)]^2}{t-\zeta}\,dt - \int_\beta\right]$$
$$-\frac{\sigma}{8\pi}\left[\int_\alpha \frac{[\omega(t)+\bar{\omega}(1/t)][\omega(t)-\bar{\omega}(1/t)]^2}{t-\zeta}\,dt - \int_\beta\right] - \frac{(1+\sigma)b}{2\pi i}\int_\gamma \frac{G(t)-\bar{G}(1/t)}{t-\zeta}\,dt$$
$$+\frac{1+\sigma}{8\pi}\left[\int_\alpha \frac{H(t)+\bar{H}(1/t)}{t-\zeta}\,dt - \int_\beta\right], \tag{3.2}$$

where

$$F(\zeta) = \int_1^\zeta [\omega(t)-a]\bar{\omega}(1/t)\,d\omega(t), \tag{3.3}$$

$$G(\zeta) = \int_1^\zeta \bar{\omega}(1/t)\,d\omega(t), \tag{3.4}$$

$$H(\zeta) = \int_1^\zeta [\omega(t)-\bar{\omega}(1/t)]^2\,d\omega(t). \tag{3.5}$$

4. The flexural moments (Ghosh, 1947a, p. 6)

$$N_1 = \iint_R \left[(1+\sigma)x'^2y' - \sigma y'^3 - x'\frac{\partial\psi_1}{\partial x'} - y'\frac{\partial\psi_1}{\partial y'}\right]dx'dy', \tag{4.1}$$

$$N_2 = \iint_R \left[\sigma x'^3 - (1+\sigma)x'y'^2 - x'\frac{\partial\psi_2}{\partial x'} - y'\frac{\partial\psi_2}{\partial y'}\right]dx'dy', \tag{4.2}$$

can be written in the form

$$N_1 = N_{11}+N_{12}+N_{13}, \quad N_2 = N_{21}+N_{22}+N_{23}, \tag{4.3}$$

where

$$N_{11} = \iint_R [(1+\sigma)(x-a)^2(y-b) - \sigma(y-b)^3 + 2(1+\sigma)axy - (1+\sigma)a(ay+bx)]\,dxdy, \tag{4.4}$$

$$N_{12} = -\iint_R \left(x\frac{\partial\Psi_1}{\partial x} + y\frac{\partial\Psi_1}{\partial y}\right)dxdy = -\int_C \Phi_1(xdx+ydy) = -\tfrac{1}{2}\boldsymbol{R}\int_C F_1\,d(z\bar{z})$$
$$= -\tfrac{1}{2}\boldsymbol{R}\int_\alpha F_1(\zeta)\,d[\omega(\zeta)\bar{\omega}(1/\zeta)] - \frac{1}{2}\int_{-1}^1 F_1(\xi)\,d[\omega(\xi)\bar{\omega}(\xi)], \tag{4.5}$$

$$N_{13} = \iint_R \left(a\frac{\partial\Psi_1}{\partial x} + b\frac{\partial\Psi_1}{\partial y}\right)dxdy = \iint_R \left(-a\frac{\partial\Phi_1}{\partial y} + b\frac{\partial\Phi_1}{\partial x}\right)dxdy$$
$$= \int_C \Psi_1(ady - bdx) = \int_C \Phi_1(adx + bdy)$$

= half the sum of the last two integrals $= \tfrac{1}{2}\boldsymbol{R}(a+ib)\int_C F_1\,d\bar{z}$

$$= \tfrac{1}{2}\boldsymbol{R}(a+ib)\left[\int_\alpha F_1(\zeta)\,d[\bar{\omega}(1/\zeta)] + \int_{-1}^1 F_1(\xi)\,d\bar{\omega}(\xi)\right], \tag{4.6}$$

$$N_{21} = \iint_R [\sigma(x-a)^3 - (1+\sigma)(x-a)(y-b)^2 + 3\sigma a(x^2-y^2) - 2(1+2\sigma)bxy$$
$$-\sigma(x^3-3xy^2) + \{(1+\sigma)b^2 - 2\sigma a^2\}x + (1+3\sigma)aby]dxdy, \quad (4.7)$$

$$N_{22} = -\iint_R \left(x\frac{\partial \Psi_2}{\partial x} + y\frac{\partial \Psi_2}{\partial y}\right)dxdy$$
$$= -\tfrac{1}{2}\boldsymbol{R}\int_\alpha F_2(\zeta)d[\omega(\zeta)\bar{\omega}(1/\zeta)] - \frac{1}{2}\int_{-1}^{1} F_2(\xi)d[\omega(\xi)\bar{\omega}(\xi)], \quad (4.8)$$

$$N_{23} = \iint \left(a\frac{\partial \Psi_2}{\partial x} + b\frac{\partial \Psi_2}{\partial y}\right)dxdy$$
$$= \tfrac{1}{2}\boldsymbol{R}(a+ib)\left[\int_\alpha F_2(\zeta)d\bar{\omega}(1/\zeta) + \int_{-1}^{1} F_2(\xi)d\bar{\omega}(\xi)\right]. \quad (4.9)$$

5. As a simple application, we take the cross-section of the beam to be the semicircular area bounded by the arc of the circle $x^2+y^2 = 1$, $y \geqq 0$ and the diameter $y = 0$, $-1 \leqq x \leqq 1$. Here $a = 0$, $b = 4/(3\pi)$, and $\omega(t) = t$. We have

$$F(\zeta) = \int_1^\zeta dt = \zeta - 1,$$

$$G(\zeta) = \int_1^\zeta \frac{dt}{t} = \log \zeta,$$

$$H(\zeta) = \int_1^\zeta (t-1/t)^2 dt = \tfrac{1}{3}\zeta^3 - 2\zeta - 1/\zeta + 8/3.$$

We have already proved in a previous paper (Ghosh, 1947b. p. 109)

$$\int_\alpha \frac{(t-1/t)^2}{t-\zeta}dt - \int_\beta = -\frac{4(\zeta^2+1)}{\zeta} + \frac{2(\zeta^2-1)^2}{\zeta^2}\log\frac{1+\zeta}{1-\zeta},$$

and it can be proved in a similar manner that

$$\int_\alpha \frac{(t+1/t)(t-1/t)^2}{t-\zeta}dt - \int = \tfrac{8}{3} - 4(\zeta^2 + 1/\zeta^2) + \frac{2(\zeta^2+1)(\zeta^2-1)^2}{\zeta^3}\log\frac{1+\zeta}{1-\zeta},$$

and

$$\int_\alpha \frac{H(t)+\bar{H}(1/t)}{t-\zeta}dt - \int_\beta = \frac{104}{9} - \frac{4}{3}\left(\zeta^2 + \frac{1}{\zeta^2}\right) + \frac{2(\zeta^6 - 9\zeta^4 + 16\zeta^3 - 9\zeta^2 + 1)}{3\zeta^3}\log\frac{1+\zeta}{1-\zeta}.$$

Further

$$\int \frac{F(t)-\bar{F}(1/t)}{t-\zeta}dt = 2\pi i\zeta,$$

$$\int_{\gamma} \frac{G(t)-\bar{G}(1/t)}{t-\zeta}\,dt = 4\pi i \log(1+\zeta);$$

$$\int_{\gamma} \frac{(t-1/t)^3}{t-\zeta}\,dt = 2\pi i(\zeta^3-3\zeta).$$

Substituting in (3.1) and (3.2), replacing ζ by z, and omitting additive constants, we get

$$F_1(z) = \tfrac{1}{4}(3+2\sigma)z+\tfrac{1}{12}(1+2\sigma)z^3+\frac{4\sigma}{3\pi^2}\left[\frac{z^2+1}{z}+\frac{(z^2-1)^2}{2z^2}\log\frac{1-z}{1+z}\right], \tag{5.1}$$

$$F_2(z) = -\frac{1-2\sigma}{6\pi}\left(z^2+\frac{1}{z^2}\right)-\frac{\sigma}{4\pi}\cdot\frac{(z^2+1)(z^2-1)^2}{z^3}\log\frac{1+z}{1-z}$$

$$+\frac{1+\sigma}{12\pi}\left[-32\log(1+z)+\frac{z^6-9z^4+16z^3-9z^2+1}{z^3}\log\frac{1+z}{1-z}\right]. \tag{5.2}$$

The flexural moments are very easily calculated. We have

$$N_{11} = -(1-6\sigma)/30-64\sigma/(27\pi^2). \tag{5.3}$$

Also

$$N_{12} = -\int_{-1}^{1} \xi F_1(\xi)d\xi.$$

Using the results

$$\int_{-1}^{1} \frac{1}{\xi}\log\frac{1+\xi}{1-\xi}\,d\xi = \frac{\pi^2}{2},\quad \int_{-1}^{1} (\xi^3-2\xi)\log\frac{1+\xi}{1-\xi}\,d\xi = -\frac{8}{3},$$

we get

$$N_{12} = -(8+\sigma)/15-16\sigma/(3\pi^2). \tag{5.4}$$

Further

$$N_{13} = \boldsymbol{R}(2i/3\pi)\left[\int_{a} F_1(\zeta)d(1/\zeta)+\int_{-1}^{1} F_1(\xi)d\xi\right].$$

As the second integral within the bracket is real,

$$N_{13} = -(2/3\pi)\boldsymbol{I}\int_{a} F_1(\zeta)d(1/\zeta) = (\tfrac{1}{2}+\tfrac{1}{3}\sigma)+64\sigma/(27\pi^2). \tag{5.5}$$

Hence

$$N_1 = -(1-7\sigma)/15-16\sigma/(3\pi^2). \tag{5.6}$$

From symmetry about the y-axis, we see at once that $N_2 = 0$.

Since the axes of x', y' are the principal axes at the centroid of the cross-section, the coordinates of the centre of flexure are (Ghosh. 1947a, p. 7)

$$x'_f = 0,\quad y'_f = -N_1/[2(1+\sigma)I_{11}] = 4(1-7\sigma+80\sigma/\pi^2)/[15\pi(1+\sigma)]$$

since

$$I_{11} = \iint_{R} x'^2\,dx'dy' = \pi/8.$$

Therefore

$$x_f = 0,\quad y_f = y'_f+4/3\pi = 8(3-\sigma+40\sigma/\pi^2)/[15\pi(1+\sigma)].$$

This result agrees with that of Stevenson (1938, p. 195) if we replace σ by $\sigma/(1-\sigma)$, the new σ being the modified Poisson's ratio of the generalized plane stress.

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DIE LORENTZTRANSFORMATION UND IHR PHYSIKALISCHER INHALT

Von

S. C. Kar

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Inhaltsübersicht : Es wird hier eine neue Ableitung der Lorentztransformation gebracht, die die Lorentzformeln von der üblichen für sie sehr einschränkenden Bedingung fast vollkommen freimacht, dass das bewegte System sich mit konstanter Geschwindigkeit gegen das ruhende bewegen muss. Es gelingt dies durch Anwendung einer Parallaxenmethode zur Messung bzw. Definition der räumlichen und zeitlichen Entfernungen eines Gegenstandes.

1. Der Lorentztransformation liegt bekanntlich das Postulat zugrunde, dass sich das bewegte Bezugsystem $(\xi, \eta, \zeta ; \tau)$ gegen das ruhende $(x, y, z ; t)$ mit konstanter Geschwindigkeit bewegen muss. Auf die Frage hin: wie lange muss diese Geschwindigkeit konstant anhalten, damit die Lorentztransformation zu recht bestehe? scheint die Antwort zunächst so lauten zu müssen: auf ewig. Eine kritische Betrachtung des Einstein'schen Gedankenganges liefert uns indessen eine genauere Antwort. Nach Einstein hat man sich, nämlich, die Zeit eines vom Beobachtungsort entfernten Gegenstandes und meines unten zu begründenden Erachtens auch dessen Abstand grundsätzlich durch einen Versuch wie den folgenden festgestellt zu denken. Der Beobachter entsendet etwa zur Zeit τ_0 einen Lichtstrahl nach dem Gegenstande und erhält ihn nach Reflexion an ihm wieder zurück etwa zur Zeit τ'_0. Dann ist die Zeit τ_G am Orte des Gegenstandes, zu welcher der Lichtstrahl ihn trifft, zu messen bzw. definieren sein durch (Einstein, 1905)

$$\tau_G = \tfrac{1}{2}(\tau_0 + \tau'_0) \tag{1}$$

und der Abstand ξ des Gegenstandes vom Beobachtungsort durch

$$\xi = \tfrac{1}{2}c(\tau'_0 - \tau_0) \tag{2}$$

wo c die Lichtgeschwindigkeit im leeren Raume ist. Allerdings nimmt Einstein an, der Abstand ξ ist einfach mit dem Längenmassstab zu messen und Gleichung (2) gestattet uns dann erst die Lichtgeschwindigkeit zu bestimmen. Voraussetzung eines Messens mit dem Massstab ist aber gewiss dies, dass der Abstand während des Messens fest bleibt, was im allgemeinen nicht zuzutreffen braucht insbesondere nämlich dann nicht, wenn der Gegenstand ein Punktereignis ist. Dagegen können wir wohl behaupten, dass das weitere Postulat Einsteins, die Lichtgeschwindigkeit c sei eine universelle Konstante, die Gleichung (2) wirklich zur Definitionsgleichung von ξ macht wohl indessen einer, die, wie es sein muss, im Einklang steht mit dem Messvorgang für die Lichtgeschwindigkeit. Die Lorentztransformation geht sodann aus der Forderung (Relativitätspostulat) hervor, die obenstehenden Definitionen von τ_G und ξ behalten ihre Geltung noch vom Gesichtspunkte eines zweiten Beobachters bei, der den ersten als mit konstanter Geschwindigkeit etwa v in der (x, ξ)-Richtung bewegt erklärt; allerdings werden dabei noch zwei Nebenhypothesen zu Hilfe gezogen, nämlich, (i) dass eine eindeutige

Korrespondenz (Forderung der Linearität) zwischen ($\xi, \eta, \zeta; \tau$) und ($x, y. z; t$) bestehen müsse und (ii) dass die vom zweiten Beobachter wahrgenommene Länge irgend eines starren bewegten in der (η, ζ)-Ebene liegenden Stabes nicht vom Sinne der Bewegungsrichtung abhänge. Die nachfolgende mathematische Herleitung (Einstein, 1905) der Transformationsformeln, welche wir nicht wiederholen wollen, zeigt, dass die in Rede stehende Geschwindigkeit für eine derartig lange Zeitspanne konstant anhalten muss, die der Lichtstrahl für den Hinweg nach und den Rückweg von dem Gegenstande benötigt. Diese Zeit fällt verschieden aus, je nachdem der Gegenstand in der Bewegungs—(ξ) oder in einer normalen (η, ζ) Richtung liegt; sie beträgt ja $\frac{2L}{c(1+v/c)}$ bzw. $\frac{2L}{c}$, wo L nach dem Urteilen des zweiten Beobachters den Abstand des Gegenstandes von der Ausgangsstelle des Lichtstrahls in dem Augenblick bedeutet, in welchem er vom Lichtstrahl getroffen wird. Wie lange die Geschwindigkeit konstant anhalten muss, hängt also jederzeit wesentlich davon ab, wie weit von der Ausgangsstelle des Lichtstrahls der Gegenstand liegt, dessen Koordinaten ($\xi, \eta, \zeta; \tau$ & $x, y, z; t$) in die Lorentzformeln eingehen; sie muss also nur dann eine auf ewig andauernd konstante sein, falls wir die Lorentzformeln für den ganzen sich ins Unendliche erstreckenden Raum in Anspruch nehmen wollen. Die vielen anderen Ableitungen der Lorentztransformation, welche in der Zwischenzeit dieser Einstein'schen gefolgt haben, ändern etwas an diesen Sachverhalt nicht, weil sie alle es zur Voraussetzung machen, dass die Geschwindigkeit eine ewig konstante sei und dabei auch keine Handhabe bieten zur Beantwortung der von uns eingangs aufgeworfenen Frage.

2. Im nachstehenden wird es unser Bestreben sein zu zeigen, dass die Sache jedoch nicht wie oben zu bestehen braucht, dass aber vielmehr der in den Lorentzformeln auftretenden Geschwindigkeit die Bedeutung einer fast momentanen Geschwindigkeit des Systems ($\xi, \eta, \zeta; \tau$) gegen das System ($x, y. z; t$) beigemessen werden kann. Um dies zu erreichen müssen wir uns die räumliche und zeitliche Entfernungen eines Gegenstandes grundsätzlich durch einen anderen Versuch als den von Einstein festgestellt denken. Wir wollen, nämlich, dass man einen, wohl etwas ideellen, doch prinzipiell brauchbaren Apparat wie den nachstehend skizzierten verwende.

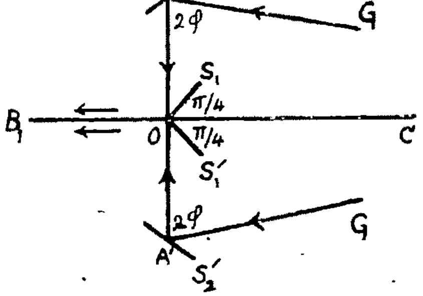

FIG. 1

Die Geraden AA', B_1OC stehen fest und zueinander normal. $OA = OA' = \lambda$, das wir die Halbbreite des Apparates nennen wollen. S_1 und S'_1 sind zwei zu O befestigte ebene Spiegel, welche mit OC den festen Winkel $\pi/4$ machen (nach oben bzw. nach unten) und etwaige von A bzw. A' ankommenden Strahlen nach B_1 reflektieren. S_2 und S'_2 sind zwei weitere ebene Spiegel, je um eine durch A bzw. A' gehende der Zeichenebene normale Achse drehbar, welche den gleichen Winkel (etwa φ) mit OC machen und bei richtiger Einstellung auf den Gegenstand nach O d.h. nach den Spiegeln S_1 bzw. S_1' reflektieren sollen. Bei richtiger Einstellung des Apparates gelangen also, wie man es nun leicht einsieht, die beiden Strahlen von A bzw. A' gleichzeitig zum Beobachter B_1 und er schliesst ohne weiteres, dass der Gegenstand G, wo die Strahlen auch gleichzeitig entstanden sind, am Schnittpunkte der Strahlen GA und GA' d.h. auf OC liegt und ferner, dass (i) die räumliche Entfernung von O nach C

$$=\lambda \tan 2\varphi = [\xi]$$

und (ii) die zeitliche Entfernung von O nach Vergangenheit

$$= \frac{\lambda}{c}(1+\sec 2\varphi) = -[\tau]$$

ist; wir postulieren wohl dabei, dass die Lichtgeschwindigkeit c eine universelle Konstante sei. Man sieht, dass es sich hier wirklich um eine Parallaxenmethode handelt. Sie stellt in dieser Form gegenüber der Einstein'schen eine *prinzipiell* einfachere Methode zur Bestimmung der räumlichen und zeitlichen Entfernungen eines Gegenstandes dar, weil sie auf einer einmaligen Beobachtung beruht und somit ein auf den Beobachtungsmoment bezogenes momentanes Bild des Punktgeschens (d.h. des Gegenstandes) liefert, wogegen bei Einstein zwei Beobachtungen (zu τ_0, τ'_0) gefordert werden. Die Gleichungen

$$\lambda \tan 2\varphi = [\xi] \tag{a}$$

$$-\frac{\lambda}{c}(1+\sec 2\varphi) = [\tau] \tag{b}$$

wollen wir deshalb zum Rang der Definitionsgleichungen von $[\xi]$, $[\tau]$ überhaupt erheben. Den Grössen λ, λ/c kommt dabei offenbar die Rolle der Längen-bzw. Zeiteinheit zu.

3. Wir führen sodann die Relativitätshypothese Einsteins ein und fordern, diese Definitionen bzw. Messungen von $[\xi]$ und $[\tau]$ müssen noch nach dem Urteilen eines zweiten Beobachters B_2 gelten, der den Apparat also auch den Beobachter B_1 als bewegt findet. Mit der Bewegung des Apparates, welche wir bis auf weiteres als gleichförmig ansetzen, geraten aber für B_2 auch die Spiegel mit in Bewegung und er muss sich mit den Gesetzen der Reflexion an bewegten Spiegeln zurechtfinden allerdings ohne jegliche Bezugnahme auf die Lorentztransformation. Dies wird aber durch das Huygens'sche Prinzip ermöglicht, welches, wie das als bekannt gelten mag, leicht ergibt

$$\sin\left(\frac{i-r}{2}\right) = \frac{V}{c}\sin\left(\frac{i+r}{2}\right) \tag{c}$$

wo i der Einfallswinkel ist, r der Reflexionswinkel und V die Komponente der Geschwindigkeit des Spiegels längs der zur Spiegelfläche nach aussen gerichteten Normale.

Des weiteren machen wir noch die Nebenhypothesen, nämlich, erstens $(\xi, \eta, \zeta; \tau)$ müssen sich in linearer Funktion von $(x, y, z; t)$ ausdrücken lassen, zweitens der bewegte Raum (ξ, η, ζ) muss nach dem Urteil von B_2 um die Bewegungsrichtung rotationssymmetrisch sein und drittens die von B_2 wahrgenommene Länge irgend eines starren bewegten Stabes muss nicht vom Sinne der Bewegungsrichtung abhängen.

4. Die Achsen (ξ, η, ζ) legen wir beziehungsweise den Geraden $(OC, A'OA)$ des Apparates und der zu deren Ebene (des Papiers) nach oben gerichteten Normale parallel und wir lassen ferner die Achsen (x, y, z) den Achsen (ξ, η, ζ) parallel laufen. Bei der Herleitung der Lorentztransformation mussen wir drei Fälle unterscheiden, je nachdem die von B_2 erklärte Bewegung des Apparates in der (I) (x, ξ)-(II) (y, η)-oder (III) (z, ζ)-Richtung stattfindet und sie gesondert behandeln.

(I) Fall der Bewegung in der (x, ξ)-Richtung

5. Die nachstehende Figur zeige an, wie für B_2 die Verhältnisse der Reflexionen an den Spiegeln durch die von ihm zugedachte gleichförmige Bewegung des Apparates geändert werden.

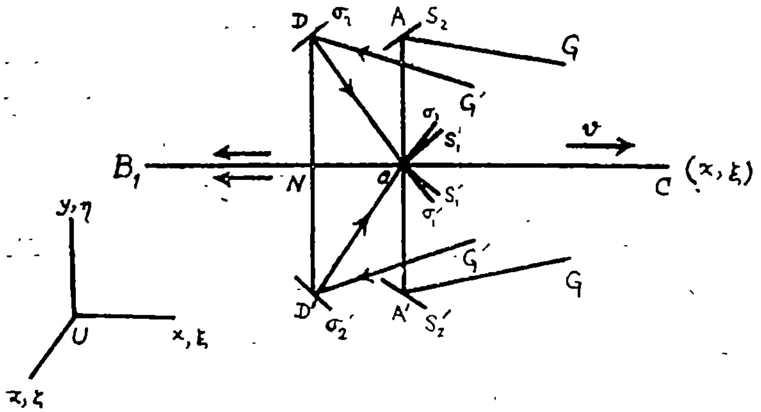

Fig. 2

Die Lage des Apparates zur Zeit, dass die Reflexionen an den (zu O) festen Spiegeln stattfinden, deuten wir in der Figur mit der Gerade $A'OA$. Da der Apparat bewegt ist, ist die Lage jedoch mit der Gerade $D'ND$ zu deuten zur früheren Zeit, zu der die Reflexionen an den (zu D', D) drehbaren Spiegeln stattfinden. Die Koordinaten von O seien für B_2 mit $(x_0, y_0, z_0; t_0)$ angegeben und ferner sei für ihn $DN = D'N = l$. Falls v die Geschwindigkeit des Apparates bezeichnet, so gilt

$$\frac{NO}{v} = \frac{DO}{c} = \frac{D'O}{c},$$

weil das Licht von D bzw. D' nach O die Gerade DO bzw. $D'O$ entlang ankommt. Falls ferner α den Winkel $\angle NDO$ order den gleichen $\angle ND'O$ bezeichnet, so ist

$$\sin\alpha = \frac{v}{c}, \qquad DO = l\sec\alpha, \qquad ON = l\tan\alpha,$$

6. Man ersieht nun folgendes: Es ist eben das Punktepaar D, D' (des Raumzeitkontinuums $x, y, z\,;\,t$), das nach dem Urteil des Beobachters B_2 vom Beobachter B_1 als das Punktepaar A, A' (des Raumzeitkontinuums $\xi, \eta, \zeta\,;\,\tau$) angeschaut wird; daher gelten für B_1 nach dem Urteil von B_2 folgende Beziehungen,

$$\left.\begin{aligned}\xi_0 &= \xi_D\\ &= \xi_{D'}\end{aligned}\right\} \qquad (d)$$

$$\left.\begin{aligned}\eta_0+\lambda &= \eta_D\\ \eta_0-\lambda &= \eta_{D'}\end{aligned}\right\} \qquad (e)$$

$$\left.\begin{aligned}\zeta_0 &= \zeta_D\\ &= \zeta_{D'}\end{aligned}\right\} \qquad (f)$$

$$\left.\begin{aligned}\tau_0-\frac{\lambda}{c} &= \tau_D\\ &= \tau_{D'}\end{aligned}\right\} \qquad (g)$$

wo $(\xi_0, \eta_0, \zeta_0\,;\,\tau_0)$, $(\xi_D, \eta_D, \zeta_D\,;\,\tau_D)$, $(\xi_{D'}, \eta_{D'}, \zeta_{D'}\,;\,\tau_{D'})$ für B_1 die Koordinaten der Punkte O, D, bzw. D' (des Raumzeitkontinuums $x, y, z\,;\,t$) sind.

Wir wollen nun $(\xi, \eta, \zeta\,;\,\tau)$ in linearer Funktion von $(x, y, z\,;\,t)$ ausdrücken, insoweit zunächst obige Beziehungen das ermöglichen. Die Beziehungen (d) lassen sich schreiben wie

$$\left.\begin{aligned}\xi(x_0 y_0 z_0 t_0) &= \xi\left(x_0-ON,\ y_0+DN,\ z_0,\ t_0-\frac{DO}{c}\right)\\ &= \xi\left(x_0-ON,\ y_0-D'N,\ z_0,\ t_0-\frac{D'O}{c}\right)\end{aligned}\right\} \qquad (d')$$

Daher folgt wegen Linearität

$$\begin{aligned}0 &= -ON\frac{\partial\xi}{\partial x_0}+DN\frac{\partial\xi}{\partial y_0}-\frac{DO}{c}\frac{\partial\xi}{\partial t_0},\\ &= -ON\frac{\partial\xi}{\partial x_0}-DN\frac{\partial\xi}{\partial y_0}-\frac{DO}{c}\frac{\partial\xi}{\partial t_0},\end{aligned}$$

oder

$$\frac{\partial\xi}{\partial y_0}=0,$$

und

$$v\frac{\partial\xi}{\partial x_0}+\frac{\partial\xi}{\partial t_0}=0.$$

Wir setzen nun

$$\frac{\partial\xi}{\partial z_0}=0$$

und berechtigen es mit folgender Überlegung. Im Falle relativer Ruhe von B_1 gegenüber B_2 würde sicher gelten

$$\xi = x.$$

Im Falle der Bewegung muss daher aus Stetigkeitsgründen (d.h. weil die Transformationsformeln für den Bewegungsfall in diejenigen für den Ruhefall stetig übergehen müssen) gelten

$$\frac{\partial \xi}{\partial x_0} \neq 0$$

und der Einfachheit halber nur noch

$$\frac{\partial \xi}{\partial t_0} \neq 0,$$

weil $\partial \xi / \partial t_0$ mit $\partial \xi / \partial x_0$ gekoppelt erscheint. Demnächst lasst sich ξ in Funktion von $(x, y, z; t)$ sofort wie

$$\xi = k_1(x - vt) + \text{konst.}$$

ausdrücken oder wie

$$\xi = k_1(x - vt) \tag{3}$$

falls ξ, x, t einen gemeinsamen Nullpunkt etwa zu U (Fig. 2) haben. Dabei ist k_1 eine noch näher zu bestimmende Konstante. Die Beziehungen (e) schreibt man ebenfalls wie

$$\left.\begin{aligned} \eta(x_0 y_0 z_0 t_0) + \lambda &= \eta\left(x_0 - ON,\ y_0 + DN,\ z_0,\ t_0 - \frac{DO}{c}\right) \\ \eta(x_0 y_0 z_0 t_0) - \lambda &= \eta\left(x_0 - ON,\ y_0 - D'N,\ z_0,\ t_0 - \frac{D'O}{c}\right) \end{aligned}\right\} \tag{e'}$$

Daher folgt

$$\lambda = -ON \frac{\partial \eta}{\partial x_0} + DN \frac{\partial \eta}{\partial y_0} - \frac{DO}{c} \frac{\partial \eta}{\partial t_0},$$

$$= ON \frac{\partial \eta}{\partial x_0} + DN \frac{\partial \eta}{\partial y_0} + \frac{DO}{c} \frac{\partial \eta}{\partial t_0},$$

oder

$$ON \frac{\partial \eta}{\partial x_0} + \frac{DO}{c} \frac{\partial \eta}{\partial t_0} = 0,$$

und

$$l \frac{\partial \eta}{\partial y_0} = \lambda.$$

Im Einklang damit und zwecks Einfachheit setzen wir

$$0 = \frac{\partial \eta}{\partial x_0} = \frac{\partial \eta}{\partial t_0} = \frac{\partial \eta}{\partial z_0}$$

Demnächst drückt sich η in Funktion von $(x, y, z; t)$ wie

$$\left.\begin{aligned} \eta &= my, \\ m &= \frac{\lambda}{l}, \end{aligned}\right\} \tag{4}$$

aus mit einem gemeinsamen Nullpunkt zu U von η, y. Aus den Beziehungen (*f*) hat man ferner

$$\left.\begin{aligned}\zeta(x_0y_0z_0t_0) &= \zeta\left(x_0-ON,\ y_0+DN,\ z_0,\ t_0-\frac{DO}{c}\right)\\ &= \zeta\left(x_0-ON,\ y_0-D'N,\ z_0,\ t_0-\frac{D'O}{c}\right)\end{aligned}\right\} \qquad (f')$$

d.h.

$$\begin{aligned}0 &= -ON\frac{\partial\zeta}{\partial x_0}+DN\frac{\partial\zeta}{\partial y_0}-\frac{DO}{c}\frac{\partial\zeta}{\partial t_0},\\ &= -ON\frac{\partial\zeta}{\partial x_0}-DN\frac{\partial\zeta}{\partial y_0}-\frac{DO}{c}\frac{\partial\zeta}{\partial t_0},\end{aligned}$$

oder

$$\frac{\partial\zeta}{\partial y_0}=0,$$

und

$$ON\frac{\partial\zeta}{\partial x_0}+\frac{DO}{c}\frac{\partial\zeta}{\partial t_0}=0.$$

Bei Ruhe von B_1 gegenüber B_2 würde offenbar gelten

$$\zeta = z.$$

Bei Bewegung muss daher

$$\frac{\partial\zeta}{\partial z_0}\neq 0$$

sein. Demnach setzen wir zwecks Einfachheit

$$0=\frac{\partial\zeta}{\partial x_0}=\frac{\partial\zeta}{\partial t_0}$$

und haben den Ausdruck

$$\zeta = k_3 z \qquad (5)$$

mit einem gemeinsamen Nullpunkt zu U von ζ, z. k_3 ist dabei eine noch näher zu bestimmende Konstante. Endlich lauten die Beziehungen (*g*) ebenfalls wie

$$\left.\begin{aligned}\tau(x_0y_0z_0t_0)-\frac{\lambda}{c} &= \tau\left(x_0-ON,\ y_0+DN,\ z_0,\ t_0-\frac{DO}{c}\right)\\ &= \tau\left(x_0-ON,\ y_0-D'N,\ z_0,\ t_0-\frac{D'O}{c}\right)\end{aligned}\right\} \qquad (g')$$

woher man hat

$$\begin{aligned}\frac{\lambda}{c} &= ON\frac{\partial\tau}{\partial x_0}-DN\frac{\partial\tau}{\partial y_0}+\frac{DO}{c}\frac{\partial\tau}{\partial t_0},\\ &= ON\frac{\partial\tau}{\partial x_0}+DN\frac{\partial\tau}{\partial y_0}+\frac{DO}{c}\frac{\partial\tau}{\partial t_0},\end{aligned}$$

oder

$$\frac{\partial\tau}{\partial y_0} = 0,$$

und

$$m \cos\alpha = v\,\frac{\partial\tau}{\partial x_0} + \frac{\partial\tau}{\partial t_0} \text{ mit } m = \frac{\lambda}{l}.$$

Wir setzen zwecks Einfachheit

$$\frac{\partial\tau}{\partial z_0} = 0$$

und alsdann drücken τ in Funktion von $(x, y, z; t)$ wie

$$\tau = k_4 x + (m \cos\alpha - v k_1) t \tag{6}$$

aus. τ, x, t haben dabei einen gemeinsamen Nullpunkt zu U und k_1 ist eine noch näher zu bestimmende Konstante.

7. Wir wollen jetzt dazu übergehen, den Vorgang der Reflexionen, welche an den zu O festen Spiegeln stattfinden, auseinanderzusetzen. Wegen Symmetrie um die Bewegungsrichtung OC genügt es nur einen Spiegel etwa S_1 ins Betracht zu ziehen. Nach dem Urteil von B_2 wird an diesem für ihn bewegten Spiegel S_1 der Strahl DO nach ON reflektiert. Gemäss dem von ihm erkannten d.h. aus dem Huygens'schen Prinzip abgeleiteten Gesetze der Reflexion an einem bewegten Spiegel kann aber eine derartige Reflexion, wie wir es sogleich finden werden, nicht statthaben, wenn der fragliche Spiegel S_1 zu OC um den von B_1 angegebenen Winkel, nämlich, $\pi/4$ geneigt ist. Wir fragen also: um welchen Winkel muss der Spiegel S_1 zu OC geneigt sein, damit die genannte Reflexion stattfinde? Es sei die Lage des Spiegels in Fig. 2 mit σ_1 gezeichdet; er mache den zu bestimmenden Winkel γ mit OC. Fur den im Auge gehaltenen Reflexionsvorgang hat man dann aus der Fig. 2 leicht

$$\text{Einfallswinkel (i)} = \gamma - \alpha,$$

$$\text{Reflexionswinkel (r)} = \frac{\pi}{2} - \gamma,$$

die Komponente (V) der Geschwindigkeit des Spiegels σ_1 längs der zur Spiegelfläche nach aussen gerichteten Normale $= -v \sin\gamma$.

Wir setzen nun diese in Gleichung (c) ein und erhalten

$$\sin\left(\gamma - \frac{\alpha}{2} - \frac{\pi}{4}\right) = -\frac{v \sin\gamma}{c} \sin\left(\frac{\pi}{4} - \frac{\alpha}{2}\right),$$

oder

$$\tan\gamma = \frac{1}{1 + v/c} \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right),$$

$$= \frac{1}{\sqrt{(1 - v^2/c^2)}},$$

mit Hilfe von : $\sin\alpha = v/c$. Der Winkel γ wird somit bestimmt. Wir überlegen nun noch folgendes. Nach B_1 lautet die analytische Gleichung des Spiegels S_1, da er mit OC den Winkel $\pi/4$ macht, wie

$$\eta-\eta_0 = \xi-\xi_0$$

Nach den Transformationsformeln (3), (4) geht sie für B_2 über in

$$y-y_0 = \frac{k_1}{m}\{(x-vt)-(x_0-vt_0)\}$$

Für ihn macht also der bewegte Spiegel σ_1 einen Winkel mit OC, dessen Tangens $= k_1/m$ ist. Der Vergleich ergibt uns somit

$$\frac{k_1}{m} = \tan\gamma = \frac{1}{\sqrt{(1-v^2/c^2)}},$$

oder

$$k_1 = \frac{m}{\sqrt{(1-v^2/c^2)}} \tag{7}$$

womit die Konstante k_1 in Abhängigkeit von m festgelegt wird.

8. Wir werden sodann die räumliche ($[x]$) und zeitliche ($[t]$) Entfernungen des Gegenstandes berechnen, wie sie B_1 nach dem Urteil von B_2 wirklich gemessen hat. Zu dem Zwecke wollen wir daher die Strahlen OD bzw. OD' über die von ihnen an den drehbaren Spiegeln erlittenen Reflexionen hinaus nach ihrem Schnittpunkte G', zu welchem nach B_2 der Gegenstand liegt, zurückverfolgen. Falls ϑ den Winkel $\angle NDG'$ oder den gleichen $\angle ND'G'$ bezeichnet, so hat man aus Fig. 2 leicht

$$[x] = l\{\tan\vartheta-\tan\alpha\},$$

$$[t] = -\frac{l}{c}\{\sec\vartheta+\sec\alpha\}.$$

Demnächst müssen wir also ϑ berechnen. Wegen Symmetrie genügt es nur die Reflexion zu betrachten, welche an einem Spiegel etwa S_2 stattfindet. Dieser hat für B_1 die Gleichung

$$\eta-(\eta_0+\lambda) = (\xi-\xi_0)\tan\varphi,$$

welche aber für B_2 nach den Transformationsformeln (3), (4) mit Hilfe von (7) in die Gleichung

$$y-(y_0+l) = \frac{\tan\varphi}{\sqrt{(1-v^2/c^2)}}\{(x-vt)-(x_0-vt_0)\}$$

übergeht. Der für B_2 bewegte Spiegel, welcher mit σ_2 bezeichnet wird, macht also einen Winkel ψ mit OC, für welchen gilt

$$\tan\psi = \tan\varphi\sec\alpha \tag{8}$$

Aus der Fig. 2 hat man nun für den Vorgang der Reflexion an σ_2

$$\text{Einfallswinkel (i)} = \vartheta - \psi,$$

$$\text{Reflexionswinkel (r)} = \psi - \alpha,$$

die Komponente (V) der Spiegelgeschwindigkeit längs der zur Spiegelfläche nach aussen gerichteten Normale $= v \sin \psi$.

Das Einsetzen von diesen in Gleichung (c) ergibt

$$\sin\left\{\frac{\vartheta-\alpha}{2} - (\psi-\alpha)\right\} = \frac{v \sin\psi}{c} \sin\frac{\vartheta-\alpha}{2},$$

oder

$$\tan\frac{\vartheta-\alpha}{2} = \tan\psi - \tan\alpha.$$

Daher hat man mit Hilfe von (8) und nach einigen einfachen Umformungen

$$\sin\vartheta = \frac{\sin 2\varphi - \sin\alpha}{1 - \sin 2\varphi \sin\alpha},$$

$$\cos\vartheta = \frac{\cos 2\varphi \cos\alpha}{1 - \sin 2\varphi \sin\alpha}.$$

Demnach hat man

$$[x] = l\left\{\frac{\sin 2\varphi - \sin\alpha}{\cos 2\varphi \cos\alpha} - \tan\alpha\right\},$$

$$[t] = -\frac{l}{c}\left\{\frac{1 - \sin 2\varphi \sin\alpha}{\cos 2\varphi \cos\alpha} + \sec\alpha\right\}.$$

Gegenüber den räumlichen und zeitlichen Abständen des Gegenstandes von O, welche B_1 nach eigener Anschauung gemessen hat, nämlich

$$[\xi] = \lambda \tan 2\varphi,$$

$$[\tau] = -\frac{\lambda}{c}(1 + \sec 2\varphi),$$

stellen $[x]$, $[t]$ also diejenigen Abstände des Gegenstandes von O dar, welche von B_1 nach der Anschauung von B_2 in der Tat gemessen werden.

9. Man sieht nun, dass $[x]$, $[t]$ und $[\xi]$ ohne weiteres die Transformationsformel (8) mit $k_1 = \dfrac{m}{\sqrt{(1-v^2/c^2)}}$ (nach (7)) befriedigen, denn man hat

$$\frac{m}{\sqrt{(1-v^2/c^2)}}\{[x] - v[t]\}$$

$$= \frac{\lambda}{l} \sec\alpha . l\left\{\frac{\sin 2\varphi - \sin\alpha}{\cos 2\varphi \cos\alpha} - \tan\alpha + \frac{v}{c}\left(\frac{1 - \sin 2\varphi \sin\alpha}{\cos 2\varphi \cos\alpha} + \sec\alpha\right)\right\}.$$

$$= \lambda \tan 2\varphi = [\xi].$$

Dieses wichtige Ergebnis besagt aber, dass jene Transformationsformel, welche aus der Bewegung des Apparates [Gleichungen (*d*), (*e*), (*f*), (*g*)] und aus den durch diese Bewegung geänderten Verhältnisse der Reflexionen gewonnen ist, sich auf die Koordinaten eines beliebig entfernten Gegenstandes ausdehnen lässt; man beachte eben, dass φ ein ganz beliebiger Winkel ist. Wir fordern nun, dass Formel (6) auch in demselben Sinne auf eine beliebige Entfernung sich erstrecken lasse, dass es nämlich gelte

$$[\tau] = k_1[x] + (m\cos\alpha - vk_4)[t] \quad .$$

d.h. $$-\frac{\lambda}{c}(1+\sec 2\varphi) = k_4 l\cos\alpha\tan 2\varphi - m\cos\alpha .\ \frac{l}{c}\left\{\frac{1-\sin 2\varphi\sin\alpha}{\cos 2\varphi\cos\alpha}+\sec\alpha\right\}$$

$$= -m\,\frac{l}{c}(1+\sec 2\varphi) + l\tan 2\varphi .\left(k_4\cos\alpha + \frac{m\sin\alpha}{c}\right).$$

Wir müssen also setzen

$$k_4\cos\alpha + \frac{m\sin\alpha}{c} = 0$$

und der Konstante k_1 den Wert $-m\tan\alpha/c$ erteilen. Die Formel (6) nimmt somit die Gestalt an

$$\tau = -\frac{m}{c}x\tan\alpha + m\left(\cos\alpha + \frac{v}{c}\tan\alpha\right)t,$$

$$= \frac{m}{\sqrt{(1-v^2/c^2)}}\left(t-\frac{vx}{c^2}\right).$$

10. Was die Konstante k_3 der Formel (5) betrifft, so setzen wir

$$k_3 = m$$

um es auszudrücken, dass der bewegte Raum (ξ, η, ζ) nach dem Urteil von, B_2 um die (x, ξ) – Richtung rotationssymmetrisch ist. Demnach lauten die Transformationsformeln wie

$$\xi = \frac{m}{\sqrt{(1-v^2/c^2)}}(x-vt), \quad \eta = my, \quad \zeta = mz, \quad \tau = \frac{m}{\sqrt{(1-v^2/c^2)}}\left(t-\frac{vx}{c^2}\right), \quad m = \frac{\lambda}{l}.$$

Die Gleichsetzung von m mit Eins erfolgt nun in fast der Weise, wie es bei Einstein geschah. Es sei nämlich der Beobachter B_2 mit einem dem vorigen gleichgebauten und parallel gestellten Messapparat, dessen Halbbreite etwa l betrage, versehen und dieser Apparat sei wieder von einem dritten Beobachter B_3, der die parallelen Achsen (x', y', z') benutze, als mit der Geschwindigkeit v in der negativen (x', x)-Richtung bewegt erklärt. Dann hat man ebenso wie oben

$$x = \frac{m'}{\sqrt{(1-v^2/c^2)}}(x'+vt'), \quad y = m'y', \quad z = m'z', \quad t = \frac{m'}{\sqrt{(1-v^2/c^2)}}\left(t'+\frac{vx'}{c^2}\right), \quad m' = \frac{l}{l'},$$

wo l' die von B_3 wahrgenommene l entsprechende Halbbreite bezeichnet. Die Hypothese, die Länge eines bewegten starren Stabes hänge nicht vom Sinne der Bewegungsrichtung ab, ergibt dann

$$\frac{l}{l'} = \frac{\lambda}{l},$$

oder

$$m' = m.$$

Daher folgt

$$x' = \frac{1}{m\sqrt{(1-v^2/c^2)}}(x-vt) = \frac{1}{m^2}\xi, \quad y' = \frac{1}{m}y = \frac{1}{m^2}\eta,$$

$$z' = \frac{1}{m}z = \frac{1}{m^2}\zeta, \quad t' = \frac{1}{m\sqrt{(1-v^2/c^2)}}\left(t-\frac{vx}{c^2}\right) = \frac{1}{m^2}\tau.$$

Man verlange nun, dass die Systeme $(\xi, \eta, \zeta; \tau)$, $(x', y', z'; t')$ identisch seien und kann also setzen

$$m = 1.$$

11. Als das wichtigste Ergebnis unserer ganzen Schlussweise wollen wir nun folgendes hervorheben. Die von B_2 erklärte Geschwindigkeit des Messapparates d.h., des Systems $(\xi, \eta, \zeta; \tau)$ gegen das System $(x, y, z; t)$ braucht bei obenstehender Herleitung der Transformationsformeln nicht eine auf ewig konstante sein. Es ist nämlich hinreichend, dass die Geschwindigkeit für eine Zeitspanne konstant anhält, welche mit

$$\frac{l}{c\sqrt{(1-v^2/c^2)}} + \varepsilon$$

anzugeben und von der Entfernung des Gegenstandes unabhängig ist. Dabei misst

$$\frac{l}{c\sqrt{(1-v^2/c^2)}}$$

die Zeit, während der der Apparat von der Lage $D'ND$ (Fig. 2) nach $A'OA$ durch die Bewegung übergeführt wird und ε bezeichnet eine nicht näher zu bestimmende, dem Null nahe Zeitgrösse, welche eventuell einem Zeitbedarf der Reflexionen an den Spiegeln Rechnung tragen soll. Jene Zeitspanne, welche also fast nur von der Halbbreite des Apparates und von dessen Geschwindigkeit abhängt, kann man demnach immerhin klein halten, falls man den Grenzfall ausschliesst, in dem v der Lichtgeschwindigkeit c unendlich nahe kommt. Es stünde in der Tat prinzipiell nichts dagegen, die Halbbreite l dem Null beliebig nahe heranreichen zu lassen, wenn nicht diesem Herannahen durch die Wellenlänge des gebrauchten Lichtes eine Grenze gezogen wäre. In diesem Sinne ist es also zu verstehen, dass der in den Transformationsformeln auftretenden Geschwindigkeit v die Bedeutung einer nahezu momentanen Geschwindigkeit des bewegten Systems $(\xi, \eta, \zeta; \tau)$ gegen das ruhende $(x, y, z; t)$ zukommen kann.

Die Transformationsformeln, wie sie oben abgeleitet worden sind, haben noch eine beschränkte Geltung; sie gelten nämlich nur in dem Falle, da s die Geschwindigkeit des

Apparates dieselbe Richtung hat, in welcher auch der Gegenstand liegt. Daher ist der Beweis davon unerlässlich, dass die Lorentzformeln auch dann gelten, wenn die Geschwindigkeit eine normale Richtung hat. Zu diesem Nachweis wollen wir jetzt also übergehen.

(II) FALL DER BEWEGUNG IN DER (y, η)-RICHTUNG

12. Die für B_2 geänderten Verhältnisse der Reflexionen an den Spiegeln seien für diesen Fall in der nachstehenden Figur angezeigt.

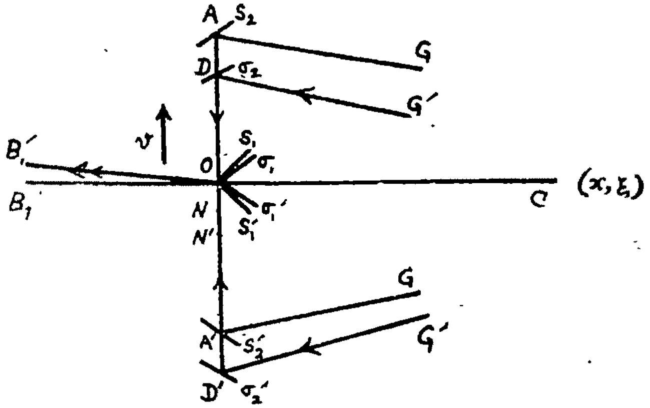

FIG. 3

Die Lage des Apparates im Augenblick, in dem die Reflexionen an den zu O festen Spiegeln stattfinden, ist mit $A'OA$ angedeutet. Zum früheren Moment der Reflexion an dem Spiegel zu D ist der Bewegung zufolge die Lage als ND, zum noch früheren Moment der Reflexion an dem Spiegel zu D' aber die Lage als $D'N'$ zu erkennen. Dabei gilt: $DN = D'N' = l$ (die von B_2 erkannte Halbbreite des Apparates). Man hat dann

$$\frac{DO}{c} = \frac{NO}{v} = \frac{DN}{c+v},$$

$$\frac{D'O}{c} = \frac{N'O}{v} = \frac{D'N'}{c-v}$$

und daher

$$DO = \frac{l}{1+v/c}, \quad D'O = \frac{l}{1-v/c}.$$

Die Koordinaten von O seien für B_2 wie früher mit $(x_0, y_0, z_0; t_0)$ angegeben.

13. Man sieht nun, dass die Beziehungen (d), (e), (f), (g) auch im vorliegenden Falle ihre Geltung haben und dass man sie an Hand von Fig. 3 zu schreiben hat wie

$$\left.\begin{aligned} \xi(x_0 y_0 z_0 t_0) &= \xi\left(x_0, y_0 + \frac{l}{1+v/c}, z_0, t_0 - \frac{l}{c(1+v/c)}\right) \\ &= \xi\left(x_0, y_0 - \frac{l}{1-v/c}, z_0, t_0 - \frac{l}{c(1-v/c)}\right) \end{aligned}\right\} \qquad (d'')$$

$$\left.\begin{aligned}\eta(x_0y_0z_0t_0)+\lambda &= \eta\left(x_0,\ y_0+\frac{l}{1+v/c},\ z_0,\ t_0-\frac{l}{c(1+v/c)}\right)\\ \eta(x_0y_0z_0t_0)-\lambda &= \eta\left(x_0,\ y_0-\frac{l}{1-v/c},\ z_0,\ t_0-\frac{l}{c(1-v/c)}\right)\end{aligned}\right\} \qquad (e'')$$

$$\left.\begin{aligned}\zeta(x_0y_0z_0t_0) &= \zeta\left(x_0,\ y_0+\frac{l}{1+v/c},\ z_0,\ t_0-\frac{l}{c(1+v/c)}\right)\\ &= \zeta\left(x_0,\ y_0-\frac{l}{1-v/c},\ z_0,\ t_0-\frac{l}{c(1-v/c)}\right)\end{aligned}\right\} \qquad (f'')$$

$$\left.\begin{aligned}\tau(x_0y_0z_0t_0)-\frac{\lambda}{c} &= \tau\left(x_0,\ y_0+\frac{l}{1+v/c},\ z_0,\ t_0-\frac{l}{c(1+v/c)}\right)\\ &= \tau\left(x_0,\ y_0-\frac{l}{1-v/c},\ z_0,\ t_0-\frac{l}{c(1-v/c)}\right)\end{aligned}\right\} \qquad (g'')$$

Daraus folgern wir die Transformationsformeln

$$\left.\begin{aligned}\xi &= k_1x,\\ \eta &= m(y-vt),\\ \zeta &= k_3z,\\ \tau &= m(t-vy/c^2)\end{aligned}\right\} \qquad (9)$$

wo $m = \lambda/l$ ist und k_1, k_3 zwei noch näher zu bestimmenden Konstanten sind. Wir wenden da das im Abschnitt 6 schon fur den vorigen Fall verwendete Schlussverfahren an unter Wahrung der Stetigkeitsbedingungen, nämlich

$$\frac{\partial\xi}{\partial x_0}\neq 0,\quad \frac{\partial\eta}{\partial y_0}\neq 0,\quad \frac{\partial\zeta}{\partial z_0}\neq 0,\quad \frac{\partial\tau}{\partial t_0}\neq 0,$$

und der Einfachheitsregel, dass sämtliche übrigen partiellen Differentialquotienten gleicher Art verschwinden ausser nur solchen, die mit jenen vier durch die Gleichungspaaren (d''), (e''), (f''), (g'') verkoppelt sind.

14. Wir wollen nun die an den zu O festen Spiegeln stattfindenden Reflexionen auseinandersetzen und merken uns zunächst folgendes: Weil B_1 nach dem Gesichtspunkte von B_2 in der (y, η)-Richtung bewegt ist, gelangen zu B_1 die an jenen Spiegeln reflektierten Strahlen nicht längs OB_1 sondern längs OB'_1 (Fig. 8), wobei OB'_1 mit OB_1 den Winkel α ($=\arcsin v/c$) nach oben macht. Wie im vorigen Falle können für B_2 auch die Lagen der Spiegeln S_1, S'_1 nicht die von B_1 angegebenen sein. Machen die bewegten Spiegel, welche wir mit σ_1, σ'_1 (Fig. 8) bezeichnen, die Winkel γ, γ' mit OC, so gilt es zunächst, diese zu berechnen.

Berechnung von γ: Am Spiegel σ_1 ist der einfallende Strahl mit DO und der reflektierte Strahl mit OB'_1 zu bezeichnen. Daher hat man in Gleichung (c) zu setzen

$$\begin{aligned}i &= \gamma,\\ r &= \tfrac{1}{2}\pi-(\gamma+\alpha),\\ V &= v\cos\gamma,\end{aligned}$$

welche Gleichung dann liefert

$$\sin\left(\gamma-\tfrac{1}{4}\pi+\tfrac{1}{2}\alpha\right)=\frac{v\cos\gamma}{c}\sin\left(\tfrac{1}{4}\pi-\tfrac{1}{2}\alpha\right)$$

oder

$$\begin{aligned}\tan\gamma&=(1+v/c)\tan\left(\tfrac{1}{4}\pi-\tfrac{1}{2}\alpha\right)\\&=\sqrt{(1-v^2/c^2)}.\end{aligned}$$

Berechnung von γ': Am Spiegel σ'_1 ist der einfallende Strahl mit $D'O$ und der reflektierte Strahl mit OB'_1 zu bezeichnen. Demnach ist in Gleichung (c) zu setzen

$$\begin{aligned}i&=\gamma',\\r&=\tfrac{1}{2}\pi-\gamma'+\alpha,\\V&=-\,v\cos\gamma'.\end{aligned}$$

Man hat dann

$$\sin\left(\gamma'-\tfrac{1}{4}\pi-\tfrac{1}{2}\alpha\right)=-\frac{v\cos\gamma'}{c}\sin\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\alpha\right)$$

oder

$$\begin{aligned}\tan\gamma'&=(1-v/c)\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\alpha\right)\\&=\sqrt{(1-v^2/c^2)}=\tan\gamma.\end{aligned}$$

Nun hat der Spiegel S_1 für B_1 die Gleichung

$$\eta-\eta_0=\xi-\xi_0$$

und der Spiegel S'_1 die Gleichung

$$\eta-\eta_0=-(\xi-\xi_0).$$

Nach den Transformationsformeln (9) gehen sie für B_2 über in

$$y-vt-(y_0-vt_0)=\frac{k_1}{m}(x-x_0)$$

bzw.

$$y-vt-(y_0-vt_0)=-\frac{k_1}{m}(x-x_0).$$

Für B_2 machen also die bewegten Spiegel σ_1, σ'_1 nach oben bzw. nach unten einen Winkel mit OC, dessen Tangens $=k_1/m$ ist. Durch Vergleich hat man demnach

$$\frac{k_1}{m}=\tan\gamma=\sqrt{\left(1-\frac{v^2}{c^2}\right)}$$

oder

$$k_1=m\sqrt{\left(1-\frac{v^2}{c^2}\right)},$$

womit die Konstante k_1 in Abhängigkeit von m ausgedrückt wird.

15. Wir werden nun die räumliche und zeitliche Entfernungen des Gegenstandes berechnen, wie sie von B_1 nach dem Urteil von B_2 in der Tat gemessen werden. Wir müssen also die Strahlen OD, OD' über die von ihnen an den Spiegeln σ_2, σ'_2 erlittenen

Reflexionen hinaus, die Geraden DG', $D'G'$ entlang nach ihrem Schnittpunkte G', welcher nach der Anschauung von B_2 eben den Ort des Gegenstandes bildet, zurückverfolgen. Bezeichnet ϑ den Winkel $\angle ODG'$ und ϑ' den Winkel $\angle OD'G'$, so ist aus dem Dreieck $DG'D'$ zu entnehmen

$$\frac{DG'}{\sin \vartheta'} = \frac{D'G'}{\sin \vartheta} = \frac{DD'}{\sin(\vartheta+\vartheta')}$$

oder, da

$$DD' = \frac{l}{1+v/c} + \frac{l}{1-v/c} = 2l \sec^2\alpha \text{ ist,}$$

$$DG' = 2l\sec^2\alpha \cdot \frac{\sin\vartheta'}{\sin(\vartheta+\vartheta')},$$

$$D'G' = 2l\sec^2\alpha \cdot \frac{\sin\vartheta}{\sin(\vartheta+\vartheta')}.$$

Demnach wird die räumliche Entfernung des Gegenstandes von O gegeben durch

$$\left.\begin{aligned} [x] &= DG'\sin\vartheta = 2l\sec^2\alpha \cdot \frac{\sin\vartheta'\sin\vartheta}{\sin(\vartheta+\vartheta')} \\ \text{und}\quad [y] &= OD - DG'\cos\vartheta = \frac{l}{1+\sin\alpha} - 2l\sec^2\alpha \cdot \frac{\sin\vartheta'\cos\vartheta}{\sin(\vartheta+\vartheta')} \end{aligned}\right\} \tag{10}$$

und die zeitliche Entfernung durch

$$\left.\begin{aligned} -[t] &= \frac{OD}{c} + \frac{DG'}{c} = \frac{l}{c(1+\sin\alpha)} + \frac{2l\sec^2\alpha}{c} \cdot \frac{\sin\vartheta'}{\sin(\vartheta+\vartheta')} \\ \text{oder}\quad &= \frac{OD'}{c} + \frac{D'D'}{c} = \frac{l}{c(1-\sin\alpha)} + \frac{2l\sec^2\alpha}{c} \cdot \frac{\sin\vartheta}{\sin(\vartheta+\vartheta')} \end{aligned}\right\} \tag{11}$$

vorbehaltlich des Nachweises, dass beide Ausdrücke rechts von (11) gleich sind. Demnächst müssen wir ϑ, ϑ' ermitteln.

Berechnung von $\sin\vartheta$, $\cos\vartheta$: Der Spiegel S_2 hat für B_1 die Gleichung

$$\eta - (\eta_0 + \lambda) = (\xi - \xi_0)\tan\varphi$$

Sie geht nach den Transformationsformeln (9), wo k_1 durch $m\cos\alpha$ zu ersetzen sein wird, in die Gleichung

$$y - vt - (y_0 - vt_0 + l) = (x - x_0)\cos\alpha\tan\varphi$$

über. Der für B_2 bewegte Spiegel σ_2 macht also einen Winkel ψ mit OC, für welchen gilt

$$\tan\psi = \cos\alpha\tan\varphi.$$

Für den Vorgang der Reflexion an σ_2 hat man also in Gleichung (c) zu setzen

$$\iota = \vartheta - \psi, \quad r = \psi, \quad V = -v\cos\psi.$$

Damit hat man

$$\sin\left(\frac{\vartheta}{2}-\psi\right) = -\frac{v\cos\psi}{c}\cdot\sin\frac{\vartheta}{2}$$

oder

$$\tan\frac{\vartheta}{2} = \frac{\tan\psi}{1+v/c} = \frac{\cos\alpha\tan\varphi}{1+\sin\alpha}$$

und daher

$$\sin\vartheta = \frac{2\left(\frac{\cos\alpha}{1+\sin\alpha}\right)\tan\varphi}{1+\left(\frac{\cos\alpha}{1+\sin\alpha}\right)^2\tan^2\varphi},$$

$$\cos\vartheta = \frac{1-\left(\frac{\cos\alpha}{1+\sin\alpha}\right)^2\tan^2\varphi}{1+\left(\frac{\cos\alpha}{1+\sin\alpha}\right)^2\tan^2\varphi}.$$

Berechnung von sin ϑ', cos ϑ': Der Spiegel S'_2 hat für B_1 die Gleichung.

$$\eta-(\eta_0+\lambda) = -(\xi-\xi_0)\tan\varphi.$$

Der Spiegel σ'_2 hat demnach für B_2 die Gleichung

$$y-vt-(y_0-vt_0+l) = -(x-x_0)\cos\alpha\tan\varphi.$$

σ'_2 macht also den gleichen Winkel ψ mit OC nach unten wie σ_2 mit OC nach oben. Für die Reflexion an σ'_2 ist also in Gleichung (c) zu setzen

$$i = \vartheta'-\psi, \quad r = \psi, \quad V = v\cos\psi.$$

Man hat dann

$$\sin\left(\frac{\vartheta'}{2}-\psi\right) = \frac{v\cos\psi}{c}\sin\frac{\vartheta'}{2}$$

oder

$$\tan\frac{\vartheta'}{2} = \frac{\tan\psi}{1-v/c} = \frac{\cos\alpha\tan\varphi}{1-\sin\alpha}$$

und daher leicht Ausdrücke für sin ϑ', cos ϑ'.

Das Einsetzen der erhaltenen Ausdrücke von sin ϑ, cos ϑ, sin ϑ', cos ϑ' in den Gleichungen (10) und (11) liefert dann ohne weiteres

$$[x] = l\sec\alpha\tan 2\varphi,$$

$$[y] = -l\tan\alpha\sec\alpha(1+\sec 2\varphi),$$

$$[t] = -\frac{l\sec^2\alpha}{c}(1+\sec 2\varphi)$$

und auch den Nachweis davon, dass die beiden Ausdrücke von $[t]$ in Gleichung (11) gleich sind.

16. Es zeigt sich nun, dass $[x]$, $[y]$ und $[t]$ den Transformationsformeln (9) mit $k_1 = m \cos\alpha$ ohne weiteres genügen, denn man hat

$$[\xi] = m \cos\alpha . [x] = \lambda \tan 2\varphi,$$

$$[\eta] = m\{[y] - v[t]\} = 0,$$

$$[\tau] = m\{[t] - \frac{v}{c}\ [y]\} = -\frac{\lambda}{c}(1 + \sec 2\varphi).$$

Daraus ziehen wir den Schluss, dass die Transformationsformeln (9) [mit $k_1 = m \cos\alpha$] an die Koordinaten eines beliebig weit entfernten Gegenstandes angewendet werden können. Die Voraussetzung der konstanten Geschwindigkeit des Systems $(\xi, \eta, \zeta\ ;\ \tau)$ gegen das System $(x, y, z\ ;\ t)$ braucht dabei aber jedoch nur für eine Zeitspanne zuzutreffen, welche fast ausschliesslich von der Halbbreite des Messapparates abhängt und auf die Entfernung des Gegenstandes gar keinen Bezug hat. Sie lässt sich mit

$$\frac{l}{c(1 - v/c)} + \varepsilon$$

angeben, wo $\frac{l}{c(1 - v/c)}$ die Zeit misst, die der Apparat benötigt um von der Lage $D'N'$ in die Lage $A'O$ überzugehen und ε einen möglichen jedenfalls sehr kleinen Zeitbedarf der Reflexionen an den Spiegeln zu decken hat. Aus dem im Abschnitt 11 geltend gemachten Grunde ist dann auch ersichtlich, dass jener Geschwindigkeit die Bedeutung einer nahezu momentanen Geschwindigkeit des bewegten Systems gegen das ruhende zukommen kann.

17. Wir setzen nun

$$k_3 = m \cos\alpha$$

weil wir es zum Ausdruck bringen wollen, dass der bewegte Raum nach dem Urteil von B_2 um die Bewegungsrichtung (y, η) rotationssymmetrisch sei. Die Transformationsformeln lauten dann

$$\left.\begin{array}{c} \xi = m\cos\alpha . x, \quad \eta = m(y - vt), \quad \zeta = m\cos\alpha . z, \\ \tau = m(t - vy/c^2), \quad m = \lambda / l \end{array}\right\} \qquad (12)$$

Um nun das Verhältnis m festzustellen wenden wir das gleiche Verfahren an, welches im vorigen Fall (I) zur Verwendung kam. Wir denken uns, nämlich, den Beobachter B_2 noch mit einem Messapparat, der dem von B_1 parallel gestellt sei und die Halbbreite l habe, versehen und dass dieser Apparat von einem dritten Beobachter B_3, der die parallelen Achsen (x', y', z') benutze, als mit der Geschwindigkeit v in der negativen (y', y)-Richtung bewegt erklärt wird. Dann hat man ebenso wie oben

$$x = m' \cos\alpha . x', \quad y = m'(y' + vt'), \quad z = m' \cos\alpha . z',$$

$$t = m'\left(t' + \frac{vy'}{c^2}\right), \quad m' = \frac{l}{l'},$$

wo l' die von B_3 erkannte Halbbreite des Apparates von B_2' ist. Die Hypothese, die Länge eines bewegten starren Stabes sei vom Sinne der Bewegungsrichtung unabhängig, ergibt

$$m' = m.$$

Daher hat man

$$(x', y', z' ; t') = \frac{1}{(m \cos \alpha)^2} (\xi, \eta, \zeta ; \tau)$$

Wir fordern nun die Identität der Systeme $(x', y', z' ; t')$ und $(\xi, \eta, \zeta ; \tau)$ und können also setzen

$$m = \sec \alpha.$$

Sonach nehmen die Transformationsformeln (12) die genaue Lorentz'sche Gestalt an.

(III) Fall der Bewegung in der (z, ζ)-Richtung

18. Die für B_2 durch die Bewegung geänderten Verhältnisse der Reflexionen an den Spiegeln seien in der nachstehenden Figur angezeigt.

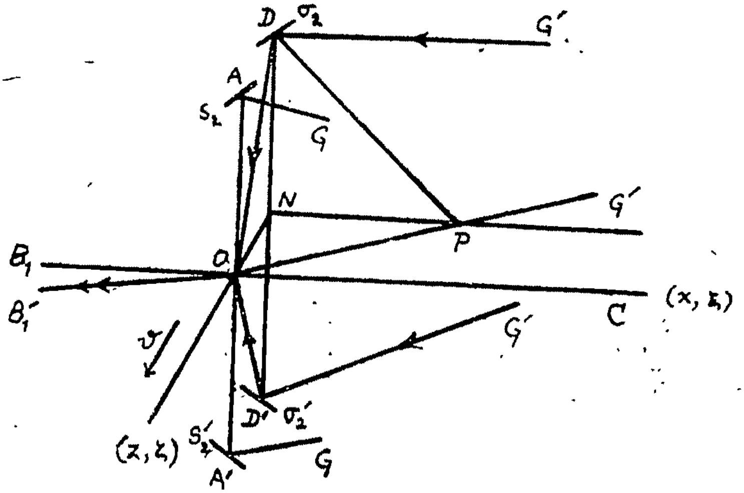

FIG. 4

Die Lage des Apparates im Augenblick, in dem die Reflexionen an den zu O festen Spiegeln stattfinden, ist mit $A'OA$ angedeutet. Zum früheren Moment der gleichzeitigen Reflexionen an den Spiegeln zu D, D' ist die Lage als $D'ND$ zu erkennen. Wir setzen $DN = D'N = l$ (die von B_2 erkannte Halbbreite des Apparates) und haben

$$\frac{D'O}{c} = \frac{DO}{c} = \frac{NO}{v} = \frac{DN}{\sqrt{(c^2 - v^2)}}$$

oder $D'O = DO = l \sec \alpha$, $NO = l \tan \alpha$. Die Koordinaten von O seien für B_2 wie früher mit $(x_0, y_0, z_0 ; t_0)$ angegeben.

19. Die Beziehungen (*d*), (*e*), (*f*), (*g*), welche auch hier ihre Geltung haben, lassen sich nun an Hand von Fig. 4 schreiben wie

$$\left.\begin{aligned}\xi(x_0y_0z_0t_0) &= \xi\left(x_0,\ y_0+l,\ z_0-l\tan\alpha,\ t_0-\frac{l\sec\alpha}{c}\right)\\ &= \xi\left(x_0,\ y_0-l,\ z_0-l\tan\alpha,\ t_0-\frac{l\sec\alpha}{c}\right)\end{aligned}\right\}\qquad (d''')$$

$$\left.\begin{aligned}\eta(x_0y_0z_0t_0)+\lambda &= \eta\left(x_0,\ y_0+l,\ z_0-l\tan\alpha,\ t_0-\frac{l\sec\alpha}{c}\right)\\ \eta(x_0y_0z_0t_0)-\lambda &= \eta\left(x_0,\ y_0-l,\ z_0-l\tan\alpha,\ t_0-\frac{l\sec\alpha}{c}\right)\end{aligned}\right\}\qquad (e''')$$

$$\left.\begin{aligned}\zeta(x_0y_0z_0t_0) &= \zeta\left(x_0,\ y_0+l,\ z_0-l\tan\alpha,\ t_0-\frac{l\sec\alpha}{c}\right)\\ &= \zeta\left(x_0,\ y_0-l,\ z_0-l\tan\alpha,\ t_0-\frac{l\sec\alpha}{c}\right)\end{aligned}\right\}\qquad (f''')$$

$$\left.\begin{aligned}\tau(x_0y_0z_0t_0)-\frac{\lambda}{c} &= \tau\left(x_0,\ y_0+l,\ z_0-l\tan\alpha,\ t_0-\frac{l\sec\alpha}{c}\right)\\ &= \tau\left(x_0,\ y_0-l,\ z_0-l\tan\alpha,\ t_0-\frac{l\sec\alpha}{c}\right)\end{aligned}\right\}\qquad (g''')$$

Daraus erhalten wir nach dem im Abschnitt 6 angewendeten Verfahren, also mit Berücksichtigung der Stetigkeitsbedingungen und der Einfachheitsregel, welche am Ende des Abschnitts 13 ausdrückliche Erwähnung finden, die Transformationsformeln

$$\left.\begin{aligned}\xi &= k_1x,\\ \eta &= my,\\ \zeta &= k_3(z-vt),\\ \tau &= k_4(z-vt)+m\cos\alpha . t,\end{aligned}\right\}\qquad (13)$$

wo $m = \lambda/l$ ist und k_1, k_3, k_1 drei noch näher zu bestimmenden Konstanten sind.

20. Wir müssen nun die Reflexionen, welche an den zu O festen Spiegeln stattfinden, auseinandersetzen; wegen einer leicht ersichtlichen Symmetrie genügt es aber nur einen Spiegel etwa S_1 bzw. σ_1 zu betrachten. Wir merken nun folgendes: da B_1 nach der Anschauung von B_2 in der (z, ζ)—Richtung bewegt ist, gelangt zu B_1 der an

jenem Spiegel reflektierte Strahl nicht längs OB_1 sondern längs OB_1', wobei OB_1' in der (x, z)—Ebene liegt und mit OB_1 den Winkel $\alpha = \text{arc.sin}(v/c)$ macht.

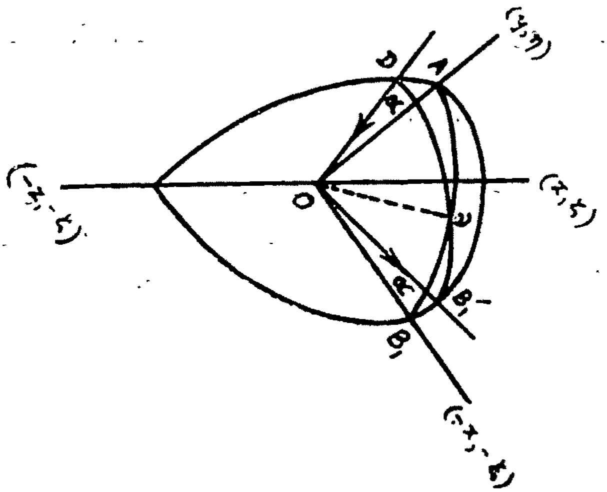

FIG. 5: Einheitskugel um O zur Veranschaulichung der Winkelverhältnisse.

Weil nun der an σ_1 reflektierte Strahl OB_1' ist und der einfallende Strahl DO, so liegt die Normale $O\nu$ (Fig. 5) zur Spiegelfläche in der Ebene DOB_1'. Der Spiegel S_1 hat für B_1 die Gleichung

$$\eta - \eta_0 = \xi - \xi_0$$

Nach den erhaltenen Transformationsformeln (18) muss also σ_1 für B_2 die Gleichung haben

$$m(y - y_0) = k_1(x - x_0).$$

Folglich muss die Normale $O\nu$ zur Spiegelfläche auch in der Ebene AOB_1 liegen. An Fig. 5 sieht man nun, dass die sphärischen Dreiecke $A\nu D$, $B_1\nu B_1'$ genau gleich sind. Die Normale $O\nu$ macht also den gleichen Winkel $\pi/4$ mit OA und OB_1. Es ergibt sich sodann

$$k_1 = m,$$

womit die Konstante k_1 auch festgelegt wird. Übrigens stimmt dieser Wert der Konstante k_1 mit der a. a. O von uns verwendeten vereinfachenden Annahme überein, dass der bewegte Raum um die Bewegungsrichtung rotationssymmetrisch ist.

21. Wir müssen sodann die zeitliche und räumliche Abstände des Gegenstandes berechnen, wie sie von B_1 nach dem Urteil von B_2 in der Tat gemessen werden. Nach B_2 liegt der Gegenstand offenbar am Schnittpunkte G' der Strahlen DG', $D'G'$, welche die auf die zu D, D' angeordneten Spiegeln σ_2, σ_2' einfallenden Strahlen darstellen entsprechend den reflektierten Strahlen DO, $D'O$. Wegen einer leicht ersichtlichen Symmetrie kann man aber den Punkt G' als Schnittpunkt der Gerade DG' mit der Ebene (x, z) betrachten. Nun lautet für B_1 die Gleichung des Spiegels S_2 wie

$$\eta-(\eta_0+\lambda) = (\xi-\xi_0)\tan\varphi$$

Nach den Formeln (13), wo jetzt k_1 durch m zu ersetzen sein wird, muss also für B_2 die Gleichung des Spiegels σ_2 lauten wie

$$y-(y_0+l) = (x-x_0)\tan\varphi$$

Die zur Spiegelfläche nach aussen gerichtete Normale DP (Fig. 4) liegt also in einer der (x, y)—Ebene parallelen Ebene und macht den Winkel φ mit DN. Es liegen ferner DG', DP, DO in der gleichen Ebene (Einfallsebene) und daher die Punkte G', P, O auf der gleichen Gerade. Weiterhin hat man für die Reflexion an σ_2 in Gleichung (c) zu setzen

$$i = \angle\, G'DP,$$

$$r = \angle\, PDO,$$

$$V = 0.$$

Gleichung (c) ergibt dann: $\angle\, G'DP = \angle\, PDO$. Wir setzen also $\angle\, G'DO = 2r$ und bezeichnen ferner $\angle\, DQG'$ mit β und $\angle\, NPO$ mit ϑ. Aus dem Dreieck $G'DO$ erhalten wir dann

$$\frac{DG'}{\sin\beta} = \frac{OG'}{\sin 2r} = \frac{OD}{\sin(\beta+2r)}.$$

oder, da $OD = l\sec\alpha$ ist.

$$DG' = \frac{l\sec\alpha\sin\beta}{\sin(\beta+2r)},\quad OG' = \frac{l\sec\alpha\sin 2r}{\sin(\beta+2r)}.$$

Demnach wird die räumliche Entfernung des Gegenstandes von O gegeben durch

$$\left.\begin{aligned} [x] &= OG'\cos\vartheta = \frac{l\sec\alpha\sin 2r\cos\vartheta}{\sin(\beta+2r)}, \\ [z] &= -OG'\sin\vartheta = -\frac{l\sec\alpha\sin 2r\sin\vartheta}{\sin(\beta+2r)}, \end{aligned}\right\} \qquad (14)$$

und die zeitliche Entfurnung durch

$$[t] = -\frac{OD}{c}-\frac{DG'}{c} = -\frac{l\sec\alpha}{c}\left\{1+\frac{\sin\beta}{\sin(\beta+2r)}\right\} \qquad (25)$$

Wir müssen nun also r, ϑ, β berechnen.

Berechnung von $\sin 2r$, $\cos 2r$. Durch Betrachtung des räumlichen Winkels $(D; O, N, P)$ hat man

$$\cos r = \cos\alpha\cos\varphi,$$

woraus

$$\cos 2r = 2\cos^2\alpha\cos^2\varphi-1,$$

$$\sin 2r = 2\cos\alpha\cos^2\varphi\sqrt{(\sin^2\alpha+\tan^2\varphi)}.$$

Berechnung von $\sin\vartheta$, $\cos\vartheta$. Man hat leicht

$$ON = l\tan\alpha,\quad NP = l\tan\varphi,$$

und

$$\sin\vartheta = \frac{ON}{OP} = \frac{\tan\alpha}{\sqrt{(\tan^2\alpha+\tan^2\varphi)}},$$

$$\cos\vartheta = \frac{NP}{OP} = \frac{\tan\varphi}{\sqrt{(\tan^2\alpha+\tan^2\varphi)}}.$$

Berchnung von $\sin\beta$, $\cos\beta$. Die Betrachtung des räumlichen Winkels $(O\,;\,P,N,D)$ liefert

$$\cos\beta = \sin\vartheta\sin\alpha = \frac{\sin\alpha\tan\alpha}{\sqrt{(\tan^2\alpha+\tan^2\varphi)}},$$

$$\sin\beta = \frac{\sqrt{(\sin^2\alpha+\tan^2\varphi)}}{\sqrt{(\tan^2\alpha+\tan^2\varphi)}}.$$

Das Einsetzen dieser Ausdrücke in Gleichungen (14) und (15) ergibt ohne weiteres

$$\begin{aligned}[x] &= l\tan 2\varphi,\\ [z] &= -l\tan\alpha.(1+\sec 2\varphi),\\ [t] &= -\frac{l\sec\alpha}{c}.(1+\sec 2\varphi).\end{aligned}$$

22. Man bestätigt nun leicht, dass $[x]$, $[z]$ und $[t]$ die Transformationsformeln (13) [mit $k_1 = m$] befriedigen, denn man hat

$$\begin{aligned}[\xi] &= m[z] = \lambda\tan 2\varphi,\\ [\zeta] &= k_3\{[z]-v[t]\} = 0,\\ [\tau] &= k_4\{[z]-v[t]\}+m\cos\alpha.[t],\\ &= -(\lambda/c)(1+\sec 2\varphi).\end{aligned}$$

Dies lässt also schliessen, dass die Transformationsformeln

$$\left.\begin{aligned}&\xi = mx,\quad \eta = my,\quad \zeta = k_3(z-vt),\\ &\tau = k_4(z-vt)+m\cos\alpha.t,\quad m = \lambda/l\end{aligned}\right\}\tag{16}$$

an die Koordinaten eines beliebig weit entfernten Gegenstandes angewendet werden können, dass aber jedoch die Geschwindigkeit des Systems $(\xi,\eta,\zeta\,;\,\tau)$ gegen das System $(x, y, z\,;\,t)$ nur für eine Zeitspanne konstant anzuhalten braucht, welche mit

$$\frac{l}{c\sqrt{\{1-(v^2/c^2)\}}}+\varepsilon$$

anzugeben ist. Dabei misst $\frac{l}{c\sqrt{\{1-(v^2/c^2)\}}}$ die Zeit, die der Apparat benötigt um von der Lage $D'ND$ in die Lage $A'OA$ überzugehen und ε deckt etwaigen Zeitbedarf der Reflexionen. Aus dem im Abschnitt 11 angeführten Grunde ist dann aber klar, dass der in den Formeln (16) enthaltenen Geschwindigkeit die Bedeutung einer fast-momentanen Geschwindigkeit des bewegten Systems gegen das ruhende beigemessen werden kann.

23. Die Konstanten k_3, k_1 und m stellen wir nun in folgender Weise fest. Wie in früheren Fällen denken wir uns den Beobachter B_2 mit einem Messapparat, der dem von B_1 parallel gestellt sei und etwa die Halbbreite l habe, versehen und dass dieser Apparat von einem dritten Beobachter B_3, der die parallelen Achsen $(x'\ y'\ z')$ verwende, als in der negativen (z', z)—Richtung mit der Geschwindigkeit v bewegt erklärt wird. Dann hat man ebenso wie oben

$$x = m'x', \quad y = m'y', \quad z = k'_3(z'+vt'),$$

$$t = k'_4(z'+vt') + m' \cos\alpha . t', \quad m' = l/l',$$

wo l' die von B_3 erkannte Halbbreite des Apparates von B_2 ist. Die Hypothese, die Länge eines bewegten starren Stabes sei vom Sinne der Bewegungsrichtung unabhängig, ergibt

$$m' = m$$

Daher hat man

$$\xi = m^2x', \quad \eta = m^2y',$$

$$\zeta = k_3\{(k'_3 - vk'_4)(z'+vt') - m \cos\alpha . vt'\},$$

$$\tau = k_1\{(k'_3 - vk'_4)(z'+vt') - m \cos\alpha . vt'\} + m \cos\alpha . \{k'_4(z'+vt') + m \cos\alpha . t'\}.$$

Wir verlangen nun, dass die Systeme $(\xi, \eta, \zeta; \tau)$, $(x', y', z'; t')$ identisch seien, dass es nämlich gelte

$$\xi = x', \quad \eta = y', \quad \zeta = z', \quad \tau = t'$$

und erhalten ohne weiteres

$$m = 1, \quad k_3 = \sec\alpha, \quad k_4 = -(v/c^2)\sec\alpha.$$

Die Transformationsformeln (16) nehmen somit die genaue Lorentzgestalt an.

24. Abschliessend wollen wir ein Paar Betrachtungen anstellen über den Begriff der Gegenständlichkeit, insoweit dieser Begriff in der Lorentztransformation sich geltend macht. Nach dem von Einstein vertretenen Gesichtspunkte kann Gegenstand der messenden Beobachtung prinzipiell der sein, der den einfallenden Lichtstrahl zurückspiegeln kann. Dagegen gehört es zum Wesen der Parallaxenmethode wie der hier von uns zugrundegelegten, dass Gegenstand der messenden Beobachtung grundsätzlich der sein soll, der gleichzeitig nach mindestens zwei Richtungen Licht entsendet, gleichviel ob das Licht eigens ausgestrahltes oder gestreutes sei. Ob der eine Begriff oder der andere *apriori* vorzuziehen ist, kann nicht gesagt werden. Wohl scheint uns aber die Parallaxenmethode der Abstandsmessung überhaupt die natürlichere.

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Referat

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ON THE TORSION AND FLEXURE OF A BEAM WHOSE CROSS-SECTION IS A QUADRANT OF A GIVEN AREA

By

S. Ghosh

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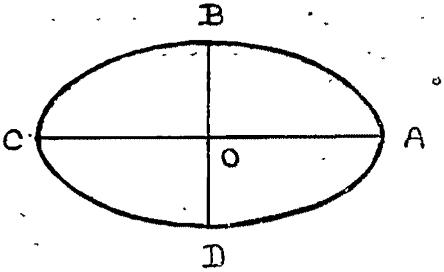
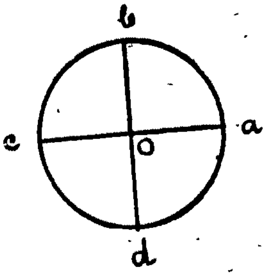
1. The method of solving the torsion-flexure problem, developed in two previous papers (Ghosh, 1947*b*, 1948) is here extended to the case of a beam whose cross-section R is a quadrant of an area R_1 which is symmetrical about two perpendicular axes and which can be mapped conformally on the unit circle by a function of simple form. The solution of the torsion-flexure problem depends, as is well-known, on the determination of a function of a complex variable which is analytic in R, and whose imaginary part takes up prescribed values on its boundary. This function, when continued analytically to the other three quadrants of R_1 by Schwarz's principle of reflection, is found to be multiple-valued in the domain R_1 with a branch point at the point of intersection of the axes of symmetry. The part of the function which is multiple-valued is then easily obtained from the value of the imaginary part of the function on the boundary of R_1. The remaining part of the function is then determined by representing R_1 conformally on the unit circle and using Schwarz's formula for an analytic function within the circle, whose imaginary part takes up given values on the circumference.

2. Let the cross-section R of the beam be the quadrant AOB of the area $ABCD$. Let OA, OB be taken as the axes of x and y respectively, and let $x = a$, $y = b$ be the coordinates of the centroid R. The coordinates of any point x, y, referred to parallel axes through the centroid are $x' = x-a$, $y' = y-b$. In the torsion-flexure problem, we are to determine three functions ψ_0, ψ_1, ψ_2 which are harmonic in R, and which satisfy the boundary conditions (Ghosh, 1947*a*, p. 4)





$$\psi_0 = \tfrac{1}{2}(x'^2+y'^2)+\text{const.}, \tag{2.1}$$

$$\psi_1 = \int_0^P [(1+\sigma)x'^2-\sigma y'^2]\,dy'+\text{const.}, \tag{2.2}$$

$$\psi_2 = -\int_0^P [(1+\sigma)y'^2-\sigma x'^2]\,dx'+\text{const.}, \tag{2.3}$$

respectively.

3. In terms of x, y, the boundary condition for ψ_0 is

$$\psi_0 = \tfrac{1}{2}(x^2+y^2)-ax-by+\text{const}.$$

Putting

$$\Psi_0 = \psi_0+ax+by-\tfrac{1}{2}(x^2-y^2), \tag{3.1}$$

we see that Ψ_0 is harmonic in R, and satisfies the boundary condition

$$\Psi_0 = y^2, \tag{3.2}$$

the additive constant being omitted.

Let $F_0(z)$ be the analytic function of z in R, whose imaginary part is Ψ_0. Then the imaginary part of the function $F_0(z)+iz^2$ is x^2 on the boundary of R, and therefore zero on OB. We can, therefore, continue $F_0(z)+iz^2$ analytically to the domain BOC by Schwarz's principle of reflection, the imaginary part of the function taking the value $-x^2$ on the boundary of BOC. Hence, in the analytic continuation of the function $F_0(z)$ to the domain BOC, its imaginary part Ψ_0 takes up the value $-(2x^2-y^2)$ on the boundary of the domain.

The imaginary part of the function $F_0(z)$ has the value y^2 on the boundary of R, and is therefore zero on OA. When this function is continued analytically, as before, to the domain AOD, its imaginary part Ψ_0 takes up the value $-y^2$ on the boundary of AOD.

Taking now the function $F_0(z)-iz^2$, we see that its imaginary part has the value $-x^2$ on the boundary of AOD, so that it is zero on OD. This function can therefore be continued analytically to the domain COD, its imaginary part taking up the value x^2 on the boundary of this domain. Hence, in the analytic continuation of $F_0(z)$ to the domain COD, Ψ_0 takes up the value $2x^2-y^2$ on the boundary of COD.

Thus we see that when $F_0(z)$ is continued analytically to the whole domain $ABCD$, it is not single-valued. As we start from a point on the lower side of OC, and describe a closed curve round O once in the positive sense, its imaginary part increases by $-4x^2$. Now, we know that the function $(2/\pi)z^2 \log z$ has a branch point at O, and as we start from a point on the lower side of OC and describe a closed curve round O once in the positive sense, its imaginary part increases by $4x^2$. Therefore the function $F_0(z)+(2/\pi)z^2 \log z$ is single-valued and analytic in the whole domain $ABCD$, and its imaginary part takes up the value

$$(1/\pi i)(z^2 \log z-\bar{z}^2 \log \bar{z})+\Psi_0, \tag{3.3}$$

on the boundary, where Ψ_0 has the values y^2, $-2x^2+y^2$, $2x^2-y^2$ and $-y^2$ on AB, BC, CD and DA respectively. If

$$z = \omega(\zeta) \tag{3.4}$$

gives the conformal representation of the domain $ABCD$, on the unit circle $abcd$ and if $F_0[\omega(\zeta)] = G_0(\zeta)$, then

$$G_0(\zeta)+(2/\pi)[\omega(\zeta)]^2 \log \omega(\zeta) \tag{3.5}$$

is analytic within the unit circle, and its imaginary part takes up the value given by the expression obtained from (3.3) by putting $z = \omega(\zeta)$ in it, on the circumference Γ. Using Schwarz's formula in its modified form (Ghosh, 1947*a*, p. 2), we get

$$G_0(\zeta)+(2/\pi)[\omega(\zeta)]^2\log\omega(\zeta)=J_1/\pi^2 i-J_2/4\pi-J_3/2\pi, \tag{3.6}$$

where

$$J_1=\int_\Gamma\frac{[\omega(t)]^2\log\omega(t)-[\bar\omega(1/t)]^2\log\bar\omega(1/t)}{t-\zeta}dt, \tag{3.7}$$

$$J_2=\int_\alpha\frac{[\omega(t)-\bar\omega(1/t)]^2}{t-\zeta}dt+\int_\beta-\int_\gamma-\int_\delta, \tag{3.8}$$

$$J_3=\int_\beta\frac{[\omega(t)+\bar\omega(1/t)]^2}{t-\zeta}dt-\int_\gamma, \tag{3.9}$$

α, β, γ, δ being the parts of Γ in the first, second, third and fourth quadrants respectively. In obtaining the integrals J_1, J_2, J_3 we have replaced $\bar t$ by $1/t$, since these integrals are taken over arcs of the unit circle.

4. To determine the flexure functions ψ_1 and ψ_2, we write the boundary conditions (2.2) and (2.3) as

$$\psi_1=[(1+\sigma)a^2-\sigma b^2]y-(1+\sigma)axy+\sigma by^2+\tfrac{1}{3}\sigma(3x^2y-y^3)+x^2y$$
$$-(1+\sigma)a\,\mathbf{I}\int_0^P\bar z\,dz-\tfrac{1}{2}(1+\sigma)\mathbf{I}\int_0^P(z^2-\bar z^2)dz, \tag{4.1}$$

$$\psi_2=-[(1+\sigma)b^2-\sigma a^2]x+(1+\sigma)bxy-\sigma ax^2+\tfrac{1}{3}\sigma(x^3-3xy^2)-xy^2$$
$$-(1+\sigma)b\,\mathbf{I}\int_0^P\bar z\,dz-\tfrac{1}{2}(1+\sigma)\mathbf{R}\int_0^P(z^2-\bar z^2)dz, \tag{4.2}$$

where **R** and **I** denote the real and the imaginary parts respectively of the integrals before which they are placed.

We see at once that

$$a\,\mathbf{I}\int_0^P\bar z\,dz+\tfrac{1}{2}\mathbf{I}\int_0^P(z^2-\bar z^2)dz,\quad b\,\mathbf{I}\int_0^P\bar z\,dz+\tfrac{1}{2}\mathbf{R}\int_0^P(z^2-\bar z^2)dz$$

vanish on OA, are respectively equal to

$$a\,\mathbf{I}\int_A^P\bar z\,dz+\tfrac{1}{2}\mathbf{I}\int_A^P(z^2-\bar z^2)dz,\quad b\,\mathbf{I}\int_A^P\bar z\,dz+\tfrac{1}{2}\mathbf{R}\int_A^P(z^2-\bar z^2)dz$$

on AB, and vanish again on BO.

Therefore, if we put

$$\Psi_1=\psi_1-[(1+\sigma)a^2-\sigma b^2]y+(1+\sigma)axy-\tfrac{1}{3}\sigma(3x^2y-y^3), \tag{4.3}$$

$$\Psi_2=\psi_2+[(1+\sigma)b^2-\sigma a^2]x-(1+\sigma)bxy+\sigma a(x^2-y^2)-\tfrac{1}{3}\sigma(x^3-3xy^2), \tag{4.4}$$

and

$$g(z)=\int_A^P\bar z\,dz,\quad h(z)=\int_A^P(z^2-\bar z^2)dz, \tag{4.5}$$

where P is a point z on AB, we see that Ψ_1 and Ψ_2 vanish on OA, have the values

$$\Psi_1=\sigma by^2+x^2y-(1+\sigma)a[g(z)-\bar g(\bar z)]/2i-(1+\sigma)[h(z)-\bar h(\bar z)]/4i, \tag{4.6}$$

$$\Psi_2=-\sigma ay^2-xy^2-(1+\sigma)b[g(z)-\bar g(\bar z)]/2i-(1+\sigma)[h(z)+\bar h(\bar z)]/4, \tag{4.7}$$

respectively on AB, and the values σby^2, $-\sigma ay^2$ respectively on BO.

Let $F_1(z)$ and $F_2(z)$ be analytic functions of z in the domain OAB, whose imaginary parts are Ψ_1 and Ψ_2 respectively. Then $F_1(z)+i\sigma bz^2$ and $F_2(z)-i\sigma az^2$ are analytic in OAB and their imaginary parts vanish on OB. Moreover, the values of these imaginary parts on AB are the same as those obtained by adding $\sigma b(x^2-y^2)$ and $-\sigma a(x^2-y^2)$ respectively to the right-hand side of (4.6) and (4.7). These functions can therefore be continued analytically, by Schwarz's principle of reflection, to the domain BOC, the values of each function at two points which are reflections of each other in OB, being complex conjugates. If z is a point in the domain BOC, or on its boundary, the corresponding point in OAB is $\bar{z}\exp(i\pi)$. Hence when $F_1(z)$ and $F_2(z)$ are continued analytically to the domain BOC, their imaginary parts Ψ_1 and Ψ_2 have the values $-2\sigma bx^2$ and $2\sigma ax^2$ respectively on OC, and the values

$$\Psi_1 = -\sigma b(2x^2-y^2)-x^2y+(1+\sigma)a\{g[\bar{z}\exp(i\pi)]-\bar{g}[z\exp(-i\pi)]\}/2i$$
$$+(1+\sigma)\{h[\bar{z}\exp(i\pi)]-\bar{h}[z\exp(-i\pi)]\}/4i, \quad (4.8)$$

$$\Psi_2 = \sigma a(2x^2-y^2)-xy^2+(1+\sigma)b\{g[\bar{z}\exp(i\pi)]-\bar{g}[z\exp(-i\pi)]\}/2i$$
$$+(1+\sigma)\{h[\bar{z}\exp(i\pi)]+\bar{h}[z\exp(-i\pi)]\}/4 \quad (4.9)$$

respectivly on BC.

Since $\Psi_1=\Psi_2=0$ on OA, $F_1(z)$ and $F_2(z)$ are, at once, continued analytically to the domain AOD. As $\bar{z}$ is the point of AOB which is the reflection in OA of the point z of AOD, we see that Ψ_1, Ψ_2 have the values $-\sigma by^2$, σay^2 respectively on OD, and the values

$$\Psi_1 = -\sigma by^2+x^2y+(1+\sigma)a[g(\bar{z})-\bar{g}(z)]/2i+(1+\sigma)[h(\bar{z})-\bar{h}(z)]/4i, \quad (4.10)$$

$$\Psi_2 = \sigma ay^2+xy^2+(1+\sigma)b[g(\bar{z})-\bar{g}(z)]/2i+(1+\sigma)[h(\bar{z})-\bar{h}(z)]/4, \quad (4.11)$$

respectively on AD.

Again the analytic functions $F_1(z)-i\sigma bz^2$ and $F_2(z)+i\sigma az^2$ have their imaginary parts zero on OD, so that they can be continued, as before, to the domain COD. If z be a point on CD, the corresponding point on AD is $\bar{z}\exp(-i\pi)$. We find that Ψ_1, Ψ_2 have the values

$$\Psi_1 = \sigma b(2x^2-y^2)-x^2y-(1+\sigma)a\{g[z\exp(i\pi)]-\bar{g}[\bar{z}\exp(-i\pi)]\}/2i$$
$$-(1+\sigma)\{h[z\exp(i\pi)]-\bar{h}[\bar{z}\exp(-i\pi)]\}/4i, \quad (4.12)$$

$$\Psi_2 = -\sigma a(2x^2-y^2)+xy^2-(1+\sigma)b\{g[z\exp(i\pi)]-\bar{g}[\bar{z}\exp(-i\pi)]\}/2i$$
$$-(1+\sigma)\{h[z\exp(i\pi)]+\bar{h}[z\exp(-i\pi)]\}/4, \quad (4.13)$$

respectively on CD. Moreover, they are equal to $2\sigma bx^2$ and $-2\sigma ax^2$ respectively on OC.

Thus we see that the analytic functions $F_1(z)$, $F_2(z)$, continued to the whole domain $ABCD$, are multiple-valued, their imaginary parts increasing by $-4\sigma bx^2$ and $4\sigma ax^2$ respectively, as we start from a point on the lower side of OC, and describe a closed curve round O once in the positive sense.

Therefore

$$F_1(z) + (2\sigma b/\pi)z^2 \log z, \qquad F_2(z) - (2\sigma a/\pi)z^2 \log z \tag{4.14}$$

are single-valued and analytic in $ABCD$, and their imaginary parts take up values

$$(\sigma b/\pi i)(z^2 \log z - \bar{z}^2 \log \bar{z}) + \Psi_1, \qquad -(\sigma a/\pi i)(z^2 \log z - \bar{z}^2 \log \bar{z}) + \Psi_2 \tag{4.15}$$

respectively, on the boundary, where Ψ_1 is given by (4.6), (4.8), (4.12), (4.10) and Ψ_2 by (4.7), (4.9), (4.13), (4.11) on AB, BC, CD, DA respectively. Proceeding as in the case of torsion and putting

$$F_1[\omega(\zeta)] = G_1(\zeta), \qquad F_2[\omega(\zeta)] = G_2(\zeta)$$

we get

$$\begin{aligned} G_1(\zeta) + (2\sigma b/\pi)[\omega(\zeta)]^2 \log \omega(\zeta) = {} & \sigma b J_1/\pi^2 i - \sigma b J_2/4\pi - \sigma b J_3/2\pi \\ & + J_4/8\pi i - (1+\sigma) a J_5/2\pi i - (1+\sigma) J_6/4\pi i, \end{aligned} \tag{4.16}$$

$$\begin{aligned} G_2(\zeta) - (2\sigma a/\pi)[\omega(\zeta)]^2 \log \omega(\zeta) = {} & -\sigma a J_1/\pi^2 i + \sigma a J_2/4\pi + \sigma a J_3/2\pi \\ & + J_4'/8\pi - (1+\sigma) b J_5/2\pi i - (1+\sigma) J_6'/4\pi, \end{aligned} \tag{4.17}$$

where J_1, J_2, J_3 are given by (3.7), (3.8), and (3.9) and

$$J_4 = \int_\alpha \frac{[\omega(t) + \bar{\omega}(1/t)]^2 [\omega(t) - \bar{\omega}(1/t)]}{t-\zeta}\,dt - \int_\beta - \int_\gamma + \int_\delta, \tag{4.18}$$

$$J_4' = \int_\alpha \frac{[\omega(t) + \bar{\omega}(1/t)][\omega(t) - \bar{\omega}(1/t)]^2}{t-\zeta}\,dt + \int_\beta - \int_\gamma - \int_\delta, \tag{4.19}$$

$$\begin{aligned} J_5 = {} & \int_\alpha \frac{g[\omega(t)] - \bar{g}[\bar{\omega}(1/t)]}{t-\zeta}\,dt - \int_\beta \frac{g[\bar{\omega}(1/t)\exp(i\pi)] - \bar{g}[\omega(t)\exp(-i\pi)]}{t-\zeta}\,dt \\ & + \int_\gamma \frac{g[\omega(t)\exp(i\pi)] - \bar{g}[\bar{\omega}(1/t)\exp(-i\pi)]}{t-\zeta}\,dt - \int_\delta \frac{g[\bar{\omega}(1/t)] - \bar{g}[\omega(t)]}{t-\zeta}\,dt, \end{aligned} \tag{4.20}$$

$$\begin{aligned} J_6' = {} & \int_\alpha \frac{h[\omega(t)] + \bar{h}[\bar{\omega}(1/t)]}{t-\zeta}\,dt - \int_\beta \frac{h[\bar{\omega}(1/t)\exp(i\pi)] + \bar{h}[\omega(t)\exp(-i\pi)]}{t-\zeta}\,dt \\ & + \int_\gamma \frac{h[\omega(t)\exp(i\pi)] + \bar{h}[\bar{\omega}(1/t)\exp(-i\pi)]}{t-\zeta}\,dt - \int_\delta \frac{h[\bar{\omega}(1/t)] + \bar{h}[\omega(t)]}{t-\zeta}\,dt, \end{aligned} \tag{4.21}$$

and J_6 is obtained from J_5 by replacing the function g by the function h.

5. The torsional and flexural moments are given by (Ghosh, 1947*a*, p. 6)

$$N_0 = \iint_R \left[x'^2 + y'^2 - x'\frac{\partial \psi_0}{\partial x'} - y'\frac{\partial \psi_0}{\partial y'}\right] dx'dy', \tag{5.1}$$

$$N_1 = \iint_R \left[(1+\sigma)x'^2 y' - \sigma y'^3 - x'\frac{\partial \psi_1}{\partial x'} - y'\frac{\partial \psi_1}{\partial y'}\right] dx'dy'. \tag{5.2}$$

$$N_2 = \iint_R \left[\sigma x'^3 - (1+\sigma)x'y'^2 - x'\frac{\partial \psi_2}{\partial x'} - y'\frac{\partial \psi_2}{\partial y'}\right] dx'dy'. \tag{5.3}$$

Substituting for ψ_0, ψ_1, ψ_2 from (3.1), (4.3), (4.4), and replacing x', y' by $x-a$, $x-b$, respectively, and simplifying, we get

$$N_0 = N_{01}+N_{02}, \quad N_1 = N_{11}+N_{12}+N_{13}, \quad N_2 = N_{21}+N_{22}+N_{23}, \tag{5.4}$$

where

$$N_{01} = 2\iint_R y^2\,dx\,dy, \tag{5.5}$$

$$N_{11} = \iint_R [(1-2\sigma)x^2y-bx^2+2\sigma by^2+2\sigma bxy-2\sigma b^3]\,dx\,dy, \tag{5.6}$$

$$N_{21} = \iint_R [-(1-2\sigma)xy^2+(1-2\sigma)ay^2-2\sigma bxy+2\sigma ab^2]\,dx\,dy, \tag{5.7}$$

$$N_{n2} = -\iint_R \left(x\frac{\partial\Psi_n}{dx}+y\frac{\partial\Psi_n}{dy}\right)dx\,dy, \quad (n = 0,\ 1,\ 2) \tag{5.8}$$

$$N_{n3} = \iint_R \left(a\frac{\partial\Psi_n}{dx}+b\frac{\partial\Psi_n}{dy}\right)dx\,dy. \qquad (n = 1,\ 2) \tag{5.9}$$

Proceeding exactly as in a previous paper (Ghosh, 1948, p. 77)

$$N_{n2} = -\tfrac{1}{2}\mathbf{R}\int F_n(z)\,d(z\bar{z}), \quad (n = 0,\ 1,\ 2)$$

$$N_{n3} = \tfrac{1}{2}\mathbf{R}\,(a+ib)\int F_n(z)\,d\bar{z}, \quad (n = 1,\ 2)$$

the line integrals being taken along the boundary of the given quadrant. Hence

$$N_{n2} = -\tfrac{1}{2}\mathbf{R}\int G_n(\zeta)\,d[\omega(\zeta)\bar{\omega}(\bar{\zeta})] \quad (n = 0,\ 1,\ 2)$$

$$N_{n3} = \tfrac{1}{2}\mathbf{R}\,(a+ib)\int G_n(\zeta)\,d[\bar{\omega}(\bar{\zeta})], \quad (n = 1,\ 2),$$

the line integrals being taken along the boundary of the quadrant of the unit circle in the ζ-plane, which is the image of the given quadrant.

Now, on the unit circle $\bar{\zeta} = 1/\zeta$, and on the real and the imaginary axes, $\bar{\zeta} = \zeta$ and $\bar{\zeta} = -\zeta$ respectively, so that

$$N_{n2} = -\tfrac{1}{2}\mathbf{R}\Big\{\int_a G_n(\zeta)\,d[\omega(\zeta)\bar{\omega}(1/\zeta)] + \int_i^0 G_n(\zeta)\,d[\omega(\zeta)\bar{\omega}(-\zeta)]$$
$$+\int_0^1 G_n(\zeta)\,d[\omega(\zeta)\bar{\omega}(\zeta)]\Big\}, \quad (n = 0,\ 1,\ 2) \tag{5.10}$$

$$N_{n3} = \tfrac{1}{2}\mathbf{R}(a+ib)\Big\{\int_a G_n(\zeta)\,d[\bar{\omega}(1/\zeta)] + \int_i^0 G_n(\zeta)\,d[\bar{\omega}(-\zeta)]$$
$$+\int_0^1 G_n(\zeta)\,d[\bar{\omega}(\zeta)]\Big\}, \quad (n = 1,\ 2). \tag{5.11}$$

The second integrals on the right-hand sides of the equations (5.10) and (5.11) are taken along the imaginary axis and the third along the real axis.

6. If the cross-section be the circular quadrant bounded by the arc of the circle $x^2+y^2=1$, $x \geqq 0$, $y \geqq 0$, and the radii $y=0$, $0 \leqq x \leqq 1$ and $x=0$, $0 \leqq y \leqq 1$, we have $\omega(t)=t$, $a=b=4/3\pi$, and

$$g(z) = \int_A^P \bar{z}\,dz = \int_1^z dz/z = \log z,$$

$$h(z) = \int_A^P (z^2-\bar{z}^2)\,dz = \int_1^z (z^2-1/z^2)\,dz = z^3/3+1/z-4/3.$$

Further,

$$J_1 = \pi i\left[1-2\left(z+\frac{1}{z}\right)+2\left(z^2+\frac{1}{z^2}\right)\log(1+z)\right],$$

$$J_2 = -4\left(z+\frac{1}{z}\right)+\left(z-\frac{1}{z}\right)^2\log\frac{(1+z)^2}{(1-z)^2},$$

$$J_3 = 2-2\left(z+\frac{1}{z}\right)+\left(z+\frac{1}{z}\right)^2\log\frac{(1+z)^2}{1+z^2},$$

$$J_4 = \frac{8i}{3}+4i\left(z^2+\frac{1}{z^2}\right)+\frac{(z^2+1)(z^4-1)}{z^3}\log\frac{(1+iz)^2}{(1-iz)^2},$$

$$J_4' = \frac{8}{3}-4\left(z^2+\frac{1}{z^2}\right)+\frac{(z^2-1)(z^4-1)}{z^3}\log\frac{(1+z)^2}{(1-z)^2},$$

$$J_5 = 2\pi i\log(1+z^2),$$

$$J_6 = -\frac{40i}{9}+\frac{4i}{3}\left(z^2+\frac{1}{z^2}\right)+\frac{(z^2-1)^3}{3z^3}\log\frac{(1+iz)^2}{(1-iz)^2},$$

$$J_6' = -\frac{40}{9}-\frac{4}{3}\left(z^2+\frac{1}{z^2}\right)+\frac{(z^2+1)^3}{3z^3}\log\frac{(1+z)^2}{(1-z)^2}-\frac{8}{3}\log\frac{(1+z^2)^2}{(1-z^2)^2}.$$

Substituting in (4.16) and (4.17), where we replace ζ by z, and omitting additive constants, we get

$$G_0(z) = \frac{1}{\pi}\left[-2z^2\log z+\log\frac{1+z^2}{1-z^2}+\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)\log(1-z^4)\right], \tag{6.1}$$

$$\begin{aligned} G_1(z) = {} & \frac{4\sigma}{3\pi^2}\left[-2z^2\log z+\log\frac{1+z^2}{1-z^2}+\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)\log(1-z^4)\right] \\ & +\frac{1}{2\pi}\left[\left(z^2+\frac{1}{z^2}\right)+\frac{(z^2+1)(z^4-1)}{2iz^3}\log\frac{1+iz}{1-iz}\right] \\ & -\frac{1+\sigma}{3\pi}\left[\left(z^2+\frac{1}{z^2}\right)+4\log(1+z^2)+\frac{(z^2-1)^3}{2iz^3}\log\frac{1+iz}{1-iz}\right], \end{aligned} \tag{6.2}$$

$$G_2(z) = -\frac{4\sigma}{3\pi^2}\left[-2z^2 \log z + \log\frac{1+z^2}{1-z^2} + \frac{1}{2}\left(z^2+\frac{1}{z^2}\right)\log(1-z^4)\right]$$

$$+\frac{1}{2\pi}\left[-\left(z^2+\frac{1}{z^2}\right)+\frac{(z^2-1)(z^4-1)}{2z^3}\log\frac{1+z}{1-z}\right]$$

$$-\frac{1+\sigma}{3\pi}\left[-\left(z^2+\frac{1}{z^2}\right)+4\log(1-z^2)+\frac{(z^2+1)^3}{2z^3}\log\frac{1+z}{1-z}\right]. \quad (6.3)$$

The expression for the complex torsion functions $\varphi_0+i\psi_0$, obtained from (3.1) and (6.1) agrees with that given by Love (1927, p. 319) and cited by him as due to Greenhill, provided we make a necessary change of axes and omit a part giving a rigid body displacement. The complex flexure functions are obtained from (4.3), (4.4), (6.2), (6.3) and have a closed form.

We now proceed to calculate the moment integrals. From considerations of symmetry, we see at once that $N_2 = -N_1$. We find easily that

$$N_{01} = \pi/8, \quad N_{11} = -(1-2\sigma)/60+\sigma/3\pi-32\sigma/27\pi^2. \quad (6.4)$$

On the unit circle, $z\bar{z} = 1$, so that

$$N_{n2} = -\tfrac{1}{2}\mathbf{R}\left[\int_1^0 G_n(z)d(-z^2)+\int_0^1 G_n(z)dz^2\right], \quad (n = 0, 1, 2) \quad (6.5)$$

$$N_{n3} = \frac{2}{3\pi}\mathbf{R}(1+i)\left[\int_1^i G_n(z)\bar{d}(1/z)+\int_i^0 G_n(z)d(-z)+\int_0^1 G_n(z)dz\right], \quad (n = 1, 2). \quad (6.6)$$

The evaluation of these integrals presents no difficulties and can be carried out in the same manner as in two previous papers (Ghosh, 1947*b*, 1948). We find

$$N_{02} = -(2/\pi)\log 2+\pi/24,$$

$$N_{12} = -(1/15\pi)[-4+28\log 2-3\pi+\sigma(8+4\log 2+40\log 2/\pi-29\pi/6)],$$

$$N_{13} = (1/3\pi)[-1+3\pi/4+\sigma(-2+\pi/2+32/9\pi)].$$

Then

$$N_0 = N_{01}+N_{02} = (\pi^2-12\log 2)/6\pi,$$

$$N_1 = N_{11}+N_{12}+N_{13} = -(1/15\pi)[1+28\log 2-13\pi/2+\sigma(13+4\log 2+40\log 2/\pi-47\pi/6)].$$

The expression for N_0 agrees with that given by Stevenson (1938, p. 193). Since $N_2 = -N_1$, the coordinates of the centre of flexure are given by (Ghosh, 1947*a*, p. 7)

$$x_f' = y_f' = -N_1/[2(1+\sigma)(I_{11}-I_{12})] = -8N_1/[(1+\sigma)(\pi-2)],$$

for

$$I_{11} = I_{22} = \pi/16-4/9\pi, \quad I_{12} = 1/8-4/9\pi.$$

Therefore

$$x_f = y_f = x_f'+4/3\pi = \frac{8}{15\pi}\cdot\frac{-4+28\log 2-4\pi+\sigma(8+4\log 2+40\log 2/\pi-16\pi/3)}{(1+\sigma)(\pi-2)},$$

This is equivalent to Stevenson's result as given by equations (11.14), (15.21), (15.34), of his paper (1938).

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RIGID ROTATION IN HYPERSPACE

By

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The object of this paper is to study by a matrix method the finite rotation of a rigid body about a point in an Euclidean n-space. Starting with the simplest rotation occurring in a plane where a skew-symmetric matrix of rank 2 is involved it is seen that a $2r$-dimensional rotation is associated with a skew-symmetric matrix of rank $2r$. The matrix method adopted depends on some new types of matrices already introduced in some of my earlier papers (Ghosh, 1940).

Representation of a General Rotation

Let the vector X to a point in the rigid body, when at rest in n-space, be represented by the skew-symmetric matrix of $(n+1)$th order

$$X = \begin{bmatrix} 0 & -x_1 & -x_2 & \dots & -x_n \\ x_1 & 0 & 0 & \dots & 0 \\ x_2 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ x_n & 0 & 0 & \dots & 0 \end{bmatrix} \tag{1}$$

where $x_1, x_2, \dots, x_n$ are the coordinates of the point referred to a rectangular system of axes fixed in space. If Λ denote the general matrix of a type, complementary to (1),

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \lambda_{11} & \lambda_{12} & \dots & \lambda_{1n} \\ 0 & \lambda_{21} & \lambda_{22} & \dots & \lambda_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \lambda_{n1} & \lambda_{n2} & \dots & \lambda_{nn} \end{bmatrix} \tag{2}$$

then after rotation of the vector X about the fixed origin, the new vector is represented by the matrix X' defined by the linear transformation

$$X' = \Delta X + X\Delta_1, \tag{3}$$

where Δ stands for the orthogonal matrix of type Λ and Δ_1 denotes the transposed of Δ; so that $\Delta\Delta_1 = \Delta_1\Delta = U_1$, U_1 being a unit matrix of type Λ.

Let Ω represent the general skew-symmetric matrix of type Λ, then the matrix Δ defined by the infinite series

$$\Delta = U_1 + \Omega + \frac{\Omega^2}{2!} + \frac{\Omega^3}{3!} + \cdots + \frac{\Omega^r}{r!} + \cdots \tag{4}$$

which we denote simply by $f(\Omega)$ satisfies the condition of orthogonality and (3) may be written in the form

$$X' = f(\Omega)X + Xf(-\Omega) = X + (\Omega X - X\Omega) + \frac{1}{2!}(\Omega^2 X + X\Omega^2) + \cdots \tag{5}$$

NOTATIONS AND FORMULAE

In what follows the scalar product of a pair of vectors X and Y is represented in the notation $\begin{Bmatrix} X \\ Y \end{Bmatrix}$, which is in conformity with the more extended type of notation

$$\begin{Bmatrix} X & Z & P & R & \cdots \\ Y & W & Q & S & \cdots \end{Bmatrix}, \tag{6}$$

involving two sets of vectors arranged in two rows enclosed within crooked brackets. This represents the determinant having for its (i, j)th element the scalar product of the ith vector in the first row and the jth vector in the second row. The properties of these determinants designated by the symbol (6) has been studied in one of my papers (Ghosh, 1936).

The following is a list of typical formulæ often used in this paper:

$$\left.\begin{aligned}
E_{11}X = XU_1, \quad X = U_1X + XU_1, \quad U_1E_{11} = 0, \\
XU_1Y = -\begin{Bmatrix} X \\ Y \end{Bmatrix}E_{11}, \quad X\Omega X = 0, \\
XY = U_1XYU_1 - \begin{Bmatrix} X \\ Y \end{Bmatrix}E_{11}, \\
XYZ = -\begin{Bmatrix} Y \\ Z \end{Bmatrix}U_1X - \begin{Bmatrix} X \\ Y \end{Bmatrix}ZU_1, \\
X^3 = -\begin{Bmatrix} X \\ X \end{Bmatrix}X, \\
XYZW = -\begin{Bmatrix} Y \\ Z \end{Bmatrix}U_1XWU_1 + \begin{Bmatrix} X \\ Y \end{Bmatrix}\begin{Bmatrix} Z \\ W \end{Bmatrix}E_{11}.
\end{aligned}\right\} \tag{7}$$

The notation E_{11} in the above represents a matrix of $(n+1)$th order with a single non-zero element 1 in the first row and the first column.

REPRESENTATION OF A PLANE ROTATION

Let A and B denote a pair of matrices of type X then $\alpha = AB - BA$, is a skew-symmetric matrix of type Λ, satisfying the characteristic equation

$$\alpha^3 + b_1^2\alpha = 0, \tag{8}$$

where

$$b_1^2 = \begin{Bmatrix} A & B \\ A & B \end{Bmatrix}. \tag{9}$$

Replacing Ω by α in (5) we have the representation of a plane rotation of X in n-space, the plane of rotation being formed by the vectors A and B.

Utilizing the relation (8), the matrix $f(\alpha)$ reduces to

$$f(\alpha) = U_1 + \frac{\sin b_1}{b_1}\alpha + \frac{1 - \cos b_1}{b_1^2}\alpha^2 \tag{10}$$

and (5) is expressed in the form

$$X' = X + \frac{\sin b_1}{b_1}(\alpha X - X\alpha) + \frac{1 - \cos b_1}{b_1^2}(\alpha^2 X + X\alpha^2). \tag{11}$$

Now

$$\alpha X - X\alpha = (AB - BA)X - X(AB - BA) = \begin{Bmatrix} A \\ X \end{Bmatrix} B - \begin{Bmatrix} B \\ X \end{Bmatrix} A \text{ by (7).}$$

Therefore, it represents a vector in the plane of A, B. Further, since $\alpha X \alpha = 0$

$$\alpha^2 X + X\alpha^2 = \alpha(\alpha X - X\alpha) - (\alpha X - X\alpha)\alpha$$

Hence $X' - X$ is a vector in the plane of A, B, which characterizes a plane rotation.

Matrices of type α satisfying (8) will be called simple skew-symmetric matrices.

It may be noted that when Ω is of rank 2 it is always reducible to the form of a simple skew-symmetric matrix.

Let Ω be represented in the form

$$\Omega = \sum_{s,\, t=2}^{n+1} \omega_{s-1,\, t-1}(e_{st} - e_{ts}), \quad t > s \tag{12}$$

where e's are matrix units and let A^i represent the matrix of type X

$$A^i = \sum_{s=2}^{n+1} \omega_{i,\, s-1}(e_{s1} - e_{1s}), \tag{13}$$

then since $\omega_{ij,\, s-1,\, t-1} = 0$, *i.e.*,

$$\omega_{ij}\omega_{s-1,\, t-1} = \omega_{i,\, s-1}\omega_{j,\, t-1} - \omega_{i,\, t-1}\omega_{j,\, s-1}$$

it follows that

$$\Omega = -\frac{A^i A^j - A^j A^i}{\omega_{ij}}, \tag{14}$$

which is in the desired form.

HIGHER DIMENSIONAL ROTATION IN n-SPACE

When Ω is of rank $2r$, the orthogonal transformation (5) represents a $2r$-dimensional rotation in n-space. It has been shown elsewhere (Ghosh, 1943) that Ω is then expressible as a sum of r simple skew-symmetric matrices of type α and hence computing $\Omega X - X\Omega$ by means of (7) it follows that it is a vector in a $2r$-dimensional space. Thus from (5) it is readily seen that $X' - X$ is a vector in a $2r$-dimensional space which characterizes a $2r$-dimensional rotation.

Let us, in particular, consider a 4-dimensional rotation, where Ω is reducible to the form

$$\Omega = AB - BA + CD - DC \tag{15}$$

A, B, C, D being all linearly independent matrices of type X. By (7) we get

$$\Omega X - X\Omega = \begin{Bmatrix} A \\ X \end{Bmatrix} B - \begin{Bmatrix} B \\ X \end{Bmatrix} A + \begin{Bmatrix} C \\ X \end{Bmatrix} D - \begin{Bmatrix} D \\ X \end{Bmatrix} C,$$

a vector in the 4-dimensional space formed by A, B, C, D. From (5) it then follows that $X' - X$ is a 4-dimensional vector.

The characteristic equation satisfied by Ω may be proved to be

$$\Omega^5 + p^2\Omega^3 + q^4\Omega = 0, \tag{16}$$

where

$$p^2 = \begin{Bmatrix} A & B \\ A & B \end{Bmatrix} + \begin{Bmatrix} C & D \\ C & D \end{Bmatrix} + 2\begin{Bmatrix} A & B \\ C & D \end{Bmatrix},$$

$$q^4 = \begin{Bmatrix} A & B & C & D \\ A & B & C & D \end{Bmatrix}.$$

Utilizing the above relation the matrix $f(\Omega)$ reduces to

$$f(\Omega) = U_1 + f_1(p,\, q)\Omega + f_2(p,\, q)\Omega^2 + f_3(p,\, q)\Omega^3 + f_4(p,\, q)\Omega^4$$

where

$$f_1(p,\, q) = 1 - \frac{q^4}{5!} + \frac{p^2q^4}{7!} - \frac{p^4q^4 - q^8}{9!} + \frac{p^6q^4 - 2p^2q^8}{11!} - \dots$$

$$f_2(p,\, q) = \frac{1}{2!} - \frac{q^4}{6!} + \frac{p^2q^4}{8!} - \frac{p^4q^4 - q^8}{10!} + \frac{p^6q^4 - 2p^2q^8}{12!} - \dots$$

$$f_3(p,\, q) = \frac{1}{3!} - \frac{p^2}{5!} + \frac{p^4 - q^4}{7!} - \frac{p^6 - 2p^2q^4}{9!} + \frac{p^8 - 3p^4q^4 + q^8}{11!} -$$

$$f_4(p,\, q) = \frac{1}{4!} - \frac{p^2}{6!} + \frac{p^4 - q^4}{8!} - \frac{p^6 - 2p^2q^4}{10!} + \frac{p^8 - 3p^4q^4 + q^8}{12!} -$$

The nature of these functions has not been further studied here.

A MATRIX IDENTITY

Let $\alpha = AB - BA$ and $\beta = CB - BC$, be a pair of simple skew-symmetric matrices. Denoting the determinants $\begin{Bmatrix} C & B \\ C & B \end{Bmatrix}$ and $\begin{Bmatrix} A & B \\ C & B \end{Bmatrix}$ by b_2^2 and b_{12} respectively, analogous to that in (9), it is easy to obtain the following relations

$$\begin{aligned} \alpha\beta\alpha &= -b_{12}\alpha, \qquad \beta\alpha\beta = -b_{12}\beta, \\ \alpha^2\beta+\beta\alpha^2 &= -b_1^2\beta - b_{12}\alpha \\ \beta^2\alpha+\alpha\beta^2 &= -b_2^2\alpha - b_{12}\beta, \\ \beta\alpha^2\beta+\beta^2\alpha^2 &= -b_1^2\beta^2 - b_{12}\beta\alpha, \\ \beta^2\alpha^2+\alpha\beta^2\alpha &= -b_2^2\alpha^2 - b_{12}\beta\alpha. \end{aligned} \tag{17}$$

Consider now the skew-symmetric matrix

$$\xi = k\alpha + l\beta + m(\beta\alpha+\alpha\beta), \tag{18}$$

where k, l, m are arbitrary scalars.

Since

$$\beta\alpha - \alpha\beta = \begin{Bmatrix} B \\ C \end{Bmatrix}\alpha - \begin{Bmatrix} A \\ B \end{Bmatrix}\beta - \begin{Bmatrix} B \\ B \end{Bmatrix}(AC-CA),$$

the above is expressible in the form of a simple skew-symmetric matrix

$$(pA+qB+rC)(p'A+q'B+r'C)-(p'A+q'B+r'C)(pA+qB+rC),$$

where p, q, r, p', q', r' are to satisfy only the three equations.

$$\left.\begin{aligned} pq'-qp' &= k + m\begin{Bmatrix} B \\ C \end{Bmatrix}, \\ rq'-qr' &= l - m\begin{Bmatrix} B \\ A \end{Bmatrix}, \\ pr'-rp' &= -m\begin{Bmatrix} B \\ B \end{Bmatrix}. \end{aligned}\right\} \tag{19}$$

Thus ξ represents a simple skew-symmetric matrix.

Squaring (18) we have

$$\begin{aligned} \xi^2 = {} & k^2\alpha^2 + kl(\beta\alpha+\alpha\beta) + l^2\beta^2 + km(\beta\alpha^2-\alpha^2\beta) \\ & + lm(\beta^2\alpha-\alpha\beta^2) + m^2(\beta\alpha\beta\alpha+\alpha\beta\alpha\beta) - m^2(\beta\alpha^2\beta+\alpha\beta^2\alpha), \end{aligned}$$

which, in virtue of the relations (17), becomes

$$\begin{aligned} \xi^2 = {} & (k^2+m^2b_2^2)\alpha^2 + (l^2+m^2b_1^3)\beta^3 + (kl+m^2b_{12})\beta\alpha + (kl-m^2b_{12})\alpha\beta \\ & + 2km\beta\alpha^2 + 2lm\beta^2\alpha + 2m^2\beta^2\alpha^2 + m(kb_1^2+lb_{12})\beta + m(lb_2^2+kb_{12})\alpha. \end{aligned}$$

Replacing $\alpha\beta$ in the above by $(k\alpha+l\beta+m\beta\alpha-\xi)/m$ from (18), we get

$$\xi^2+\frac{1}{m}(kl-m^2b_{12})\xi = a^2\alpha^2+b^2\beta^2+2kl\beta\alpha+2km\beta\alpha^2+2lm\beta^2\alpha+2m^2\beta^2\alpha^2+\frac{l}{m}a^2\alpha+\frac{k}{m}b^2\beta, \quad (20)$$

where the symbols a^2 and b^2 denote respectively

$$k^2+m^2b_2^2 \quad \text{and} \quad l^2+m^2b_1^2.$$

It is now obvious that the relation (20) yields the identity

$$U_1+\frac{2}{m}(kl-m^2b_{12})\xi+2\xi^2 = \left(U_1+\frac{2k}{m}b^2\beta+2b^2\beta^2\right)\left(U_1+\frac{2l}{m}a^2\alpha+2a^2\alpha^2\right) \quad (21)$$

provided k, l, m be connected by the equation

$$(k^2+m^2b_2^2)(l^2+m^2b_1^2) = m^2. \quad (22)$$

The above equation yields, in general, four values of m corresponding to a given set of values for k, l. Let the roots be denoted by m', m'', $-m'$, $-m''$, then to determine the pair m', m'', for instance we have the quadratic equation

$$b_1b_2m^2-m\{1-(kb_1-lb_2)^2\}^{\frac{1}{2}}+kl = 0 \quad (23)$$

whence

$$m' = \frac{\{1-(kb_1-lb_2)^2\}^{\frac{1}{2}}+\{1-(kb_1+lb_2)^2\}^{\frac{1}{2}}}{2b_1b_2}, \quad (24)$$

$$m'' = \frac{\{1-(kb_1-lb_2)^2\}^{\frac{1}{2}}-\{1-(kb_1+lb_2)^2\}^{\frac{1}{2}}}{2b_1b_2}. \quad (25)$$

It may be noted that taking all possible signs of k, l, m into consideration there are eight distinct values of ξ admitting fatorization in the form (21). For reality of the roots of m, when k, l are real some obvious conditions are to be satisfied.

COMPOSITION OF PLANE ROTATIONS WHEN THE RESULTANT ROTATION IS ALSO PLANE

Let k and l be expressed in terms of a pair of angular magnitudes θ and φ, so that

$$kb_1 = \frac{\cos\varphi+\cos\theta}{2} = \cos\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}, \quad (26)$$

$$lb_2 = \frac{\cos\varphi-\cos\theta}{2} = \sin\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2}. \quad (28)$$

Then from (24) and (25) we have

$$b_1b_2m' = (\sin\theta+\sin\varphi)/2 = \sin\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}, \quad (28)$$

$$b_1b_2m'' = (\sin\theta-\sin\varphi)/2 = \cos\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2}. \quad (29)$$

The values of a^2 and b^2 corresponding to $m = m'$ are respectively $\cos^2\frac{1}{2}(\theta-\varphi)/b_1^2$ and $\sin^2\frac{1}{2}(\theta+\varphi)/b_2^2$, whence those corresponding to $m = m''$ are to be obtained by changing φ into $-\varphi$.

Substituting now the values k, l, m' obtained from (26), (27). (28) in (21) and simplifying we get

$$U_1+2\left(\cos\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2}-\sin\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}\cos\psi\right)\xi+2\xi^2$$

$$=\left(U_1+\frac{\sin(\theta+\varphi)}{b_2}\beta+\frac{2\sin^2\frac{1}{2}(\theta+\varphi)}{b_2^{\,2}}\beta^2\right)\left(U_1+\frac{\sin(\theta-\varphi)}{b_1}\alpha+\frac{2\cos^2\frac{1}{2}(\theta-\varphi)}{b_1^{\,2}}\alpha^2\right), \quad (30)$$

where $\xi = k\alpha+l\beta+m'(\beta\alpha-\alpha\beta)$ and $\cos\psi = b_{12}/b_1b_2$, ψ being the angle between the planes of α and β.

Putting

$$\bar{\alpha} = \frac{\cos\frac{1}{2}(\theta-\varphi)}{b_1}\alpha \quad \text{and} \quad \bar{\beta} = \frac{\sin\frac{1}{2}(\theta+\varphi)}{b_2}\beta$$

(30) takes the form

$$U_1+2\{\cos\tfrac{1}{2}(\theta+\varphi)\sin\tfrac{1}{2}(\theta-\varphi)-\sin\tfrac{1}{2}(\theta+\varphi)\cos\tfrac{1}{2}(\theta-\varphi)\cos\psi\}\xi+2\xi^2$$

$$=\{U_1+2\cos\tfrac{1}{2}(\theta+\varphi)\bar{\beta}+2\bar{\beta}^2\}\{U_1+2\sin\tfrac{1}{2}(\theta-\varphi)\bar{\alpha}+2\bar{\alpha}^2\},$$

where

$$\xi = \cos\tfrac{1}{2}(\theta+\varphi)\bar{\alpha}+\sin\tfrac{1}{2}(\theta-\varphi)\bar{\beta}+(\bar{\beta}\bar{\alpha}-\bar{\alpha}\bar{\beta}),$$

Reducing further by putting $\theta+\varphi = b_2$ and $\theta-\varphi = \pi-b_1$ we get

$$U_1+2(\cos\tfrac{1}{2}b_2\cos\tfrac{1}{2}b_1+\sin\tfrac{1}{2}b_2\sin\tfrac{1}{2}b_1\cos\psi)\xi+2\xi^2$$

$$=(U_1+2\cos\tfrac{1}{2}b_2\bar{\beta}+2\bar{\beta}^2)(U_1+2\cos\tfrac{1}{2}b_1\bar{\alpha}+2\bar{\alpha}^2), \quad (31)$$

where

$$\bar{\alpha} = \frac{\sin\frac{1}{2}b_1}{b_1}\alpha, \qquad \bar{\beta} = \frac{\sin\frac{1}{2}b_2}{b_2}\beta,$$

and

$$\xi = \cos\tfrac{1}{2}b_2\bar{\alpha}+\cos\tfrac{1}{2}b_1\bar{\beta}+(\bar{\beta}\bar{\alpha}-\bar{\alpha}\bar{\beta}).$$

Referring now to (10) which is equivalent to

$$U_1+2\cos\tfrac{1}{2}b_1\bar{\alpha}+2\bar{\alpha}^2,$$

where

$$\bar{\alpha} = \frac{\sin\frac{1}{2}b_1}{b_1}\alpha$$

it follows that (31) leads to the composition formula of two plane rotations occurring in a three dimensional space immersed in an Euclidean n-space. In a recent paper (Schwerdtfeger, 1945) has treated the problem of paramatrization of the three-dimensional rotation group by an application of Matrix Calculus (Schwerdtfeger, 1945).

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CANONICAL CO-ORDINATES IN GENERAL RELATIVITY

By

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Summary. The gravitational metric is derived in canonical coordinates involving directly the values of the twenty Riemann-Christoffel components at the origin. The derivation implies solving a set of hundred linear equations. The field equations and several tensor conditions are expressed in the neighbourhood of the origin in terms of the constant components. The nature of the geodesics and the velocity of light in the neighbourhood of the space-time origin are discussed.

1. Canonical Coordinates

At any point of the space-time field of gravitation it is possible (Einstein, Lorentz and others, 1923) to express the metric,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{1}$$

in the form,

$$ds^2 = -dx^2 - dy^2 - dz^2 + d\tau^2. \tag{2}$$

Either (2) is not valid in a finite neighbourhood of the point or the coordinates x, y, z, τ are quasi-coordinates. For a rigorous investigation of the neighbourhood we have to confine ourselves only to Gaussian transformations and transform (1) into a form which reduces at the given point, taken as a space-time origin, to (2). The scalar invariants defined at a point are not altered by the Gaussian transformations nor does a tensor like R_{hijk}, which is non-zero at a point in a gravitational field, vanish by virtue of such a transformation. There are fourteen scalar invariants of the Riemann-Christoffel tensor R_{hijk} which remain unaltered at any space-time point where (1) is transformed into (2). It is not therefore correct to say that gravitation can thus be transformed away at a point. Only a Riemannian coordinate system would help us to investigate the significance of the transformation. The canonical coordinate system seems to be best suited for the purpose because the twenty essential components of R_{hijk} are explicitly involved in it. We have not seen anywhere the gravitational metric expressed in canonical coordinates and hence proceed to obtain it first. For a Riemannian coordinate system (Eisenhart, 1926a), at the origin,

$$\frac{\partial g_{\mu\nu}}{\partial x^\sigma} = 0, \tag{3}$$

that is, all the first order partial derivatives of $g_{\mu\nu}$ vanish. The hundred second order partial derivatives of $g_{\mu\nu}$ satisfy the hundred relations (Eddington, 1924)

$$\left.\begin{aligned} \frac{\partial}{\partial x^\nu}\Gamma^\epsilon_{\mu\sigma} - \frac{\partial}{\partial x^\sigma}\Gamma^\epsilon_{\mu\nu} &= R^\epsilon_{\mu\nu\sigma}, \\ \frac{\partial}{\partial x^\sigma}\Gamma^\epsilon_{\mu\nu} + \frac{\partial}{\partial x^\nu}\Gamma^\epsilon_{\mu\sigma} + \frac{\partial}{\partial x^\mu}\Gamma^\epsilon_{\nu\sigma} &= 0, \end{aligned}\right\} \tag{4}$$

when the Riemannian system is canonical. If x'^{μ} denotes x, y, z, τ for $\mu = 1, 2, 3, 4$ respectively we have the formula,

$$x^{\sigma} = g^{\sigma}_{\mu}x'^{\mu} - \tfrac{1}{2}(\Gamma^{\sigma}_{\mu\nu})_0 x'^{\mu}x'^{\nu} - \frac{1}{18}\left(\frac{\partial}{\partial x^{\mu}}\Gamma^{\sigma}_{\nu\sigma} + \frac{\partial}{\partial x^{\nu}}\Gamma^{\sigma}_{\mu\sigma} + \frac{\partial}{\partial x^{\sigma}}\Gamma^{\sigma}_{\mu\nu}\right)_0 x'^{\mu}x'^{\nu}x'^{\sigma}, \tag{5}$$

in the usual notation, it being understood that the formulae (3) and (4) are indentities in the x'-system.

The new metric tensor $g'_{\mu\nu}$ for which the dashes may safely be omitted is given near the origin at (x, y, z, τ) by the equations,

$$g_{\mu\nu} = (g_{\mu\nu})_0 + \frac{1}{2!}\left[\left(\frac{\partial g_{\mu\nu}}{\partial x^2}\right)_0 x^2 + \left(\frac{\partial g_{\mu\nu}}{\partial y^2}\right)_0 y^2 + 2\left(\frac{\partial g_{\mu\nu}}{\partial x \partial y}\right)_0 xy + \cdots\right]. \tag{6}$$

Without any loss of generality, one can take

$$(g_{\mu\nu})_0 = 0, \quad \mu \neq \nu$$
$$(g_{11})_0 = (g_{22})_0 = (g_{33})_0 = (-g_{44})_0 = -1. \tag{7}$$

Having solved the equations (4) we get

$$\begin{aligned}
g_{11} &= -1 - \tfrac{1}{3}(ay^2 + bz^2 + c\tau^2 - 2gyz - 2hy\tau - 2iz\tau),\\
g_{22} &= -1 - \tfrac{1}{3}(ax^2 + dz^2 + e\tau^2 - 2jxz - 2kx\tau - 2lz\tau),\\
g_{33} &= -1 - \tfrac{1}{3}(bx^2 + dy^2 + f\tau^2 - 2mxy - 2nx\tau - 2oy\tau),\\
g_{44} &= 1 - \tfrac{1}{3}(cx^2 + ey^2 + fz^2 - 2pxy - 2qxz - 2ryz),\\
g_{12} &= \tfrac{1}{3}(mz^2 + p\tau^2 + axy - gxz - hx\tau - jyz - ky\tau - (2t+s)z\tau),\\
g_{13} &= \tfrac{1}{3}(jy^2 + q\tau^2 + bxz - gxy - ix\tau - myz - mz\tau + (t-s)y\tau),\\
g_{14} &= \tfrac{1}{3}(ky^2 + nz^2 + cx\tau - hxy - ixz - py\tau - qz\tau + (t+2s)yz),\\
g_{23} &= \tfrac{1}{3}(gx^2 + r\tau^2 + dyz - jxy - mxz - ly\tau - oz\tau + (t+2s)x\tau),\\
g_{24} &= \tfrac{1}{3}(hx^2 + oz^2 + ey\tau - kxy - px\tau - lyz - rz\tau + (t-s)xz),\\
g_{34} &= \tfrac{1}{3}(ix^2 + ly^2 + fz\tau - nxz - qx\tau - oyz - ry\tau - (2t+s)xy).
\end{aligned} \tag{8}$$

In the above, the twenty essential components of R_{hijk} where

$$R_{hijk} = \frac{1}{2}\left(\frac{\partial^2 g_{hk}}{\partial x^i \partial x^j} + \frac{\partial^2 g_{ij}}{\partial x^h \partial x^k} - \frac{\partial^2 g_{hj}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^h \partial x^j}\right), \tag{9}$$

appear through

$$\begin{aligned}
&R_{1212} = a, && R_{1313} = b, && R_{1414} = c,\\
&R_{2323} = d, && R_{2424} = e, && R_{3434} = f,\\
&R_{2113} = g, && R_{2114} = h, && R_{3114} = i,\\
&R_{1223} = j, && R_{1224} = k, && R_{3224} = l,\\
&R_{1332} = m, && R_{1334} = n, && R_{2334} = o,\\
&R_{1442} = p, && R_{1443} = q, && R_{2443} = r,\\
&R_{1234} = s, && R_{1423} = t.
\end{aligned} \tag{10}$$

As we are interested only in a neighbourhood of the origin, powers above the second of x, y, z, τ may be neglected throughout. The structure of the terms in the expansions of $g_{\mu\nu}$ deserves notice. It is clear that

$$\frac{\partial g_{44}}{\partial x} = -2\frac{\partial g_{14}}{\partial \tau}, \quad \frac{\partial g_{22}}{\partial z} = -2\frac{\partial g_{23}}{\partial y}, \text{ etc.} \tag{11}$$

if only the terms explicitly stated in the expansions are considered.

2. The Field Equations

At the origin, the field equations,

$$g^{hk}R_{hijk} \equiv R_{ij} = \lambda g_{ij}, \tag{12}$$

run as follows:

$$\begin{aligned} a+b-c &= -\lambda, \\ a+d-e &= -\lambda, \\ b+d-f &= -\lambda, \\ c+e+f &= \lambda; \end{aligned} \tag{13}$$

$$\begin{gathered} -m+p = 0, \quad -j+q = 0, \quad -g+r = 0, \\ -k-n = 0, \quad -h-o = 0, \quad -i-l = 0. \end{gathered} \tag{14}$$

We have from (13)

$$a+f = b+e = c+d = 0 \tag{15}$$

and

$$R \equiv g^{ij}R_{ij} = 2(c+e+f-a-b-d) = 4\lambda. \tag{16}$$

It is interesting to note that, as a result of the field equations,

$$|g_{\mu\nu}| = -1 + \frac{\lambda}{8}(x^2+y^2+z^2-\tau^2). \tag{17}$$

The conformal curvature tensor C_{hijk} (Eisenhart, 1926b) has only ten essential components. They are given by

$$\begin{aligned} C_{2113} &= C_{2443} = \tfrac{1}{2}(g+r), \\ C_{1332} &= C_{1442} = \tfrac{1}{2}(p+m), \\ C_{1223} &= C_{1443} = \tfrac{1}{2}(j+q), \\ C_{1224} &= -C_{1334} = \tfrac{1}{2}(k-n), \\ C_{2114} &= -C_{2334} = \tfrac{1}{2}(h-o), \\ C_{3114} &= -C_{3224} = \tfrac{1}{2}(i-l), \\ C_{1234} &= s, \ C_{1423} = t; \end{aligned} \tag{18}$$

$$\begin{aligned} C_{1212} &= -C_{3434} = \tfrac{1}{3}(a-f) + \tfrac{1}{6}(c+e-b-d), \\ C_{1313} &= -C_{2424} = \tfrac{1}{3}(b-e) + \tfrac{1}{6}(c+f-a-d), \\ C_{1414} &= -C_{2323} = \tfrac{1}{3}(c-d) + \tfrac{1}{6}(a+b-e-f). \end{aligned}$$

It may be verified that.

$$g^{hk}C_{hijk} = 0. \tag{19}$$

At the origin, the ten equations,

$$C_{hijk} = 0, \tag{20}$$

supplemented by the ten, viz.,

$$R_{ij} = 0, \tag{21}$$

make all the twenty constants zero.

For a space-time of constant curvature k_0, we have

$$R_{hijk} = k_0(g_{hj}g_{ik} - g_{hk}g_{ij}), \tag{22}$$

which implies

$$a = b = d = -c = -e = -f = k_0, \tag{23}$$

the remaining fourteen constants being zero.

For a material distribution given by the perfect fluid tensor,

$$T_{ij} = (\rho + p')v_i v_j - p' g_{ij}, \quad g^{ij}v_i v_j = 1, \tag{24}$$

where ρ is the density, p' the pressure and v_i the flow vector, the field equations

$$R_{ij} - \tfrac{1}{2}Rg_{ij} + \lambda g_{ij} = -8\pi T_{ij}, \tag{25}$$

give

$$\left.\begin{aligned} a+b-c+\tfrac{1}{2}R-\lambda &= -8\pi[(\rho+p')v_1^2+p'], \\ a+d-e+\tfrac{1}{2}R-\lambda &= -8\pi[(\rho+p')v_2^2+p'], \\ b+d-f+\tfrac{1}{2}R-\lambda &= -8\pi[(\rho+p')v_3^2+p'], \\ c+e+f-\tfrac{1}{2}R+\lambda &= -8\pi[(\rho+p')v_4^2-p'], \end{aligned}\right\} \tag{26}$$

$$\left.\begin{aligned} -m+p &= -8\pi(\rho+p')v_1v_2, \\ -j+q &= -8\pi(\rho+p')v_1v_3, \\ -k-n &= -8\pi(\rho+p')v_1v_4, \\ -g+r &= -8\pi(\rho+p')v_2v_3, \\ -h-o &= -8\pi(\rho+p')v_2v_4, \\ -i-l &= -8\pi(\rho+p')v_3v_4. \end{aligned}\right\} \tag{27}$$

It is clear that

$$2(c+e+f-a-b-d) = 8\pi(\rho-3p')+4\lambda \tag{28}$$

and that

$$(p-m)(i+l) = (q-j)(h+o) = (r-g)(k+n). \tag{29}$$

One can also establish several equations like

$$2(a+b+c+d+e+f) = (i+l)\left[\frac{r-g}{h+o}+\frac{h+o}{r-g}\right]+(k+n)\left[\frac{q-j}{i+l}+\frac{i+l}{q-j}\right]$$

$$+(h+o)\left[\frac{p-m}{k+n}+\frac{k+n}{p-m}\right]. \tag{30}$$

We have also

$$8\pi(\rho+p') = \frac{4(p-m)(k+n)}{h+o} + 2(c-a-b-e-f+d) \tag{31}$$

$$= \tfrac{1}{2}N/(a+b+c+d+e+f), \tag{32}$$

where

$$N = 4[(p-m)^2+(q-j)^2+(r-g)^2+(k+n)^2+(h+o)^2+(i+l)^2]$$
$$-4(ce+ef+fc)-4(ab+bd+da)-4(ac+ae+bc+bf+de+df). \tag{33}$$

Separate expressions for ρ, p', v_1^2 etc. can be obtained easily from the above results.

Thus, even if the metric form (2) at the origin is the same as that of special relativity we see how the local latent curvatures affect the metric in the neighbourhood and the distribution of matter at the origin.

3. The Geodesics

The equations of the geodesics are

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0. \tag{34}$$

If we use τ as the independent variable there are three equations of the form,

$$\frac{d^2x}{d\tau^2} + \Gamma^1_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} - \frac{dx}{d\tau}\Gamma^4_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0. \tag{35}$$

For a test particle which is at (0, 0, 0) at $\tau = 0$ and has

$$dx/d\tau = dy/d\tau = dz/d\tau = 0, \tag{36}$$

at $\tau = 0$, we get

$$\begin{aligned} \frac{d^2x}{d\tau^2} - \tfrac{2}{3}(cx-py-qz) &= 0, \\ \frac{d^2y}{d\tau^2} - \tfrac{2}{3}(cy-px-rz) &= 0, \\ \frac{d^2z}{d\tau^2} - \tfrac{2}{3}(fz-qx-ry) &= 0. \end{aligned} \tag{37}$$

In free space, if $R_{\mu\nu} = 0$ or $R_{\mu\nu} = \lambda g_{\mu\nu}$ with $\lambda > 0$, we have from (13) that

$$c+e+f \geq 0. \tag{38}$$

But the equations (37) cannot describe a small oscillatory motion (Datta Majumdar, 1946) unless

$$c < 0, \quad e < 0, \quad f < 0, \tag{39}$$

which contradicts (38). In a region of space occupied by matter, however, the conditions of small oscillatory motion may be fulfilled (Narlikar, 1936).

The velocity of light in the coordinates of (2) is unity at the origin in all spatial directions. But in the neighbourhood of the space-time origin, we have

$$g_{11}dx^2+2g_{14}dx\,d\tau+g_{44}d\tau^2=0, \tag{40}$$

for a pulse of light. Here, if the velocity of light as affected by the gravitational field is not to exceed unity, we must have

$$\begin{aligned}(ay^2+bz^2+c\tau^2-2gyz-2hy\tau-2iz\tau)+(cx^2+ey^2+fz^2-2pxy-2qxz-2ryz)\\ \geq 2\,|\,ky^2+nz^2-hxy-ixz+cx\tau+(t+2s)yz-py\tau-qz\tau\,|,\end{aligned} \tag{41}$$

which requires along with other conditions

$$a+e>0, \quad b+f>0. \tag{42}$$

But (42) and (15) cannot be harmonised. Hence the coordinate velocity of light, in the neighbourhood of the origin, will necessarily exceed unity in some direction. This can probably be avoided by modifying (7).

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NOTE ON SOME THEOREMS ON THE HÖLDER AND CESÀRO MEANS

By

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1. The present note is mainly concerned with a problem on the Hölder and Cesàro means, similar to one considered by Kuttner (1939) for the Riesz and Cesàro sums. In the last section a problem of the same nature has also been considered for two means corresponding to two equivalent Hausdorff methods. The results mostly came out as offshoots of a paper by the author (Basu (1)) in which the problem of total relative strength of the Hölder and Cesàro methods has been considered. It will be observed that the positiveness or otherwise of the transformations involved plays a similar role in this problem and the problems with which we are now concerned (*cf.* Basu (2), Introduction).

2. We require some properties of Hausdorff transformations (Hausdorff, 1921). The *Hausdorff mean* σ_n of a *real* sequence $\{s_n\}$ or of a series $\sum_{\nu=0}^{\infty} u_\nu$, written in short as $\sum u_n$, for which $s_n = \sum_{\nu=0}^{n} u_\nu$, is defined by the transformation

$$\text{H}: \qquad \sigma_n = \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k} \mu_k . s_k \quad (n = 0, 1, 2, \dots) \tag{1}$$

where $\{\mu_n\}$ is a given *factor sequence* and

$$\Delta^0 \mu_k = \mu_k, \quad \Delta^n \mu_k = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} \mu_{k+\nu} \quad (n \geq 1).$$

In case $\sigma_n \to l$*, $\sum u_n$ is said to be *summable* to l by the Hausdorff method $H \sim \mu_n$ (See Rogosinski, 1942, p. 166). We shall suppose, in what follows, that

$$\mu_n \neq 0 \qquad (n = 0, 1, 2, \dots),$$

so that the *inverse* transformation for (1) is

$$s_n = \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k} \left(\frac{1}{\mu_k}\right) \cdot \sigma_k \qquad (n = 0, 1, 2, \dots). \tag{2}$$

Let C_n^α and H_n^α respectively denote the Cesàro and Hölder means of order $\alpha > -1$ for the series $\sum u_n$ and C^α and H^α the corresponding methods. Then as special cases of (1), we have

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* Unless otherwise stated, l denotes a finite number.

$$C^{\alpha} \sim \mu_n = \frac{1}{A_n^{\alpha}} = \frac{1}{\binom{n+\alpha}{n}}, \qquad H^{\alpha} \sim \mu_n' = \frac{1}{(n+1)^{\alpha}}, \tag{3}$$

the two methods being identical for $\alpha = 0, 1$. Also

$$C_n^{\alpha} = \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k} \left(\frac{\mu_k}{\mu'_k}\right). H_k^{\alpha} \tag{4}$$

and

$$H_n^{\alpha} = \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k} \left(\frac{\mu'_k}{\mu_k}\right) \cdot C_k^{\alpha}. \tag{5}$$

It is well known that for $\alpha > -1$,

$$C^{\alpha} \approx H^{\alpha}, \tag{6}$$

i.e., the two methods of summability are equivalent, in the sense that $C_n^{\alpha} \to l$ implies $H_n^{\alpha} \to l$ and conversely. The transformation (1) is said to be *regular* if $s_n \to l$ implies $\sigma_n \to l$. We note that, in virtue of (2) and (6), the transformations (4) and (5) are both regular.

A necessary condition of regularity of (1) is that (See Hausdorff p. 81 or Dienes, p. 385)

$$\sum_{k=0}^{n} \binom{n}{k} |\Delta^{n-k} \mu_k| \leq Q^* \text{ for all } n. \tag{7}$$

We first prove the following theorem.

Theorem 1. *For $r > 0$, $\alpha > -1$,*

$$C_n^{\alpha} = O(n^r)$$

implies

$$H_n^{\alpha} = O(n^r),$$

and conversely.

Proof. Suppose that for $r > 0$, $\alpha > -1$,

$$C_n^{\alpha} = O(n^r).$$

then

$$|C_n^{\alpha}| < Qn^r \text{ for all } n > 0.$$

Hence from (5),

$$\begin{aligned} |H_n^{\alpha}| &\leq \left|\Delta^n \left(\frac{\mu_0'}{\mu_0}\right)\right| |C_0^{\alpha}| + \sum_{k=1}^{n} \binom{n}{k} \left|\Delta^{n-k} \left(\frac{\mu_k'}{\mu_k}\right)\right| |C_k^{\alpha}| \\ &< \left|\Delta^n \left(\frac{\mu_0'}{\mu_0}\right)\right| |C_0^{\alpha}| + Qn^r \sum_{k=1}^{n} \binom{n}{k} \left|\Delta^{n-k} \left(\frac{\mu_k'}{\mu_k}\right)\right| \\ &< Qn^r \text{ by (7) ;} \end{aligned}$$

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* As here, Q will be used throughout to denote an unspecified positive constant, *i.e.* a number independent of the parameter of the argument and need not be the same in each occurence.

and the first part of the theorem follows. The converse can be similarly proved.

3. Next, our problem is to examine whether Theorem 1 continues to hold when O is replaced by O_L. We shall also consider the allied problem, whether for a given $\alpha > -1$, $C_n^\alpha \geq 0$ for all n implies $H_n^\alpha \geq 0$ for all n and conversely.

The following theorems will be proved.

Theorem 2. *For a given $\alpha > -1$, let $X(\alpha)$ denote the proposition "If $C_n^\alpha \geq 0$ for all n, then $H_n^\alpha \geq 0$ for all n," and let $Y(\alpha)$ denote the converse proposition. Let $X'(\alpha)$ denote the proposition "For a given $r > 0$, $C_n^\alpha = O_L(n^r)$ implies $H_n^\alpha = O_L(n^r)$", and let $Y'(\alpha)$ denote the converse proposition. Then*

(i) $X(\alpha)$ and $X'(\alpha)$ are equivalent, in the sense that they are both true for the same values of α;

(ii) $Y(\alpha)$ and $Y'(\alpha)$ are equivalent.

Theorem 3. *(i) For $0 \leq \alpha \leq 1$, $Y'(\alpha)$ is true but $X'(\alpha)$ is false;*

(ii) for $\alpha > 1$, $X'(\alpha)$ is true but not $Y'(\alpha)$;

(iii) for $-1 < \alpha < 0$, $X'(\alpha)$ is true but $Y'(\alpha)$ is not.

Theorem 4. *(i) For $0 \leq \alpha \leq 1$, $Y(\alpha)$ is true but $X(\alpha)$ is false;*

(ii) for $\alpha > 1$, $X(\alpha)$ is true but $Y(\alpha)$ is not;

(iii) for $-1 < \alpha < 0$, $X(\alpha)$ is true but not $Y(\alpha)$.

The proofs of the different parts of Theorem 3 are similar and so also is the case for Theorem 4. Accordingly, we take up only the proofs of Theorems 3(*i*) and 4(*i*) Theorem 2 follows from Theorems 3 and 4.

We require the following lemma (See Basu (1)). The present lemma follows from Lemma 1 and Theorems 1, 2 and 3 of this paper).

Lemma *(i) For $0 < \alpha < 1$, the sequence $\{\mu_n/\mu_n'\}$ as defined in (3) is totally monotone, that is,*

$$\Delta^n\left(\frac{\mu_k}{\mu_k'}\right) \geq 0 \qquad (n,\ k = 0, 1, 2, \ldots),$$

but the reciprocal sequence is not totally monotone;

(ii) for $\alpha > 1$, the sequence $\{\mu_n'/\mu_n\}$ is totally monotone but not $\{\mu_n/\mu_n'\}$;

(iii) for $-1 < \alpha < 0$, the sequence $\{\mu_n'/\mu_n\}$ is totally monotone but the reciprocal sequence is not.

Proof of Theorem 3(*i*). Suppose that for $r > 0$. $0 < \alpha < 1$,

$$H_n^\alpha = O_L(n^r).$$

Then

$$H_n^\alpha > -Qn^r \text{ for all } n > 0.$$

Hence from (4),

$$C_n^\alpha = \Delta^n\left(\frac{\mu_0}{\mu_0'}\right) H_0^\alpha + \sum_{k=1}^{n} \binom{n}{k} \Delta^{n-k}\left(\frac{\mu_k}{\mu_k'}\right) \cdot H_k^\alpha$$

$$> \Delta^n\left(\frac{\mu_0}{\mu_0'}\right) H_0^\alpha - Qn^r \sum_{k=1}^{n} \binom{n}{k} \Delta^{n-k}\left(\frac{\mu_k}{\mu_k'}\right) \quad \text{by Lemma } (i)$$

$$> -Qn^r \quad \text{in virtue of (7).}$$

This proves the first part of Theorem 3(*i*).

To prove the second part we observe that in virtue of Lemma (*i*), for $0 < \alpha < 1$, all the coefficients of C_k^α in (5) are not ≥ 0. In fact, the coefficient of $C_{n-1}^\alpha < 0$ $(n \geq 1)$; for

$$\Delta\left(\frac{\mu'_{n-1}}{\mu_{n-1}}\right) = A_{n-1}^\alpha \left\{\frac{1}{n^\alpha} - \frac{n+\alpha}{n} \frac{1}{(n+1)\alpha}\right\},$$

and since $A_{n-1}^\alpha > 0$, and for $0 < \alpha < 1$,

$$1 + \frac{\alpha}{n} > \left(1 + \frac{1}{n}\right)^\alpha,$$

is follows that

$$\Delta\left(\frac{\mu'_{n-1}}{\mu_{n-1}}\right) < 0 \qquad (n = 1, 2, 3, \ldots). \tag{8}$$

On using the transformation (5), given any positive function $f(n)$ of $n > 0$, however large, in virtue of (8), we can define a series $\sum u_n$ inductively with the aid of (2), such that for it

$$C_n^\alpha \geq 0 \qquad (n = 0, 1, 2, \ldots),$$

but

$$H_{2n}^\alpha < -f(2n) \qquad (n = 1, 2, 3, \ldots).$$

In particular, we can take $f(n)$ such that

$$H_n^\alpha \neq O_L(n^r) \qquad \text{for a given } r > 0.$$

The proof of the second part of Theorem 3(*i*) is now complete.

Proof of Theorem 4(*i*). For $0 < \alpha < 1$, all the coefficients of H_k^α in (4) are ≥ 0. Hence when $H_n^\alpha \geq 0$ for all n, it follows that $C_n^\alpha \geq 0$ for all n.

Again, in (5), $\{\mu_n'/\mu_n\}$ is now not totally monotone. Hence

$$\Delta^p\left(\frac{\mu'_m}{\mu_m}\right) < 0 \qquad \text{for some } p \text{ and } m.$$

We define $\sum u_n$ inductively such that

$$C_n^\alpha = 0 \qquad (n \neq m)$$
$$= 1 \qquad (n = m).$$

Then from (5),

$$H^{\alpha}_{m+p} = \binom{m+p}{p}\Delta^{p}\left(\frac{\mu'_{m}}{\mu_{m}}\right) < 0.$$

This completes the proof of Theorem 4(*i*).

4. It may be observed that in the proof of Theorem 1 no special properties of C^{α}_{n} and H^{α}_{n} have been used. Accordingly, the result of this theorem may be generalized for any two equivalent Hausdorff methods with non-vanishing moment sequences. Thus suppose $A \sim a_n$, $B \sim b_n$ are two Hausdorff methods, such that

$$a_n \neq 0, \quad b_n \neq 0 \qquad (n = 0, 1, 2, \ldots);$$

and

$$A \approx B.$$

Then we have the theorem:

Theorem I. *If A_n and B_n be the Hausdorff means corresponding to A and B respectively, then*

$$A_n = O(n^r) \qquad (r > 0)$$

implies

$$B_n = O(n^r),$$

and conversely.

Further, *if A totally includes B, i.e.* if for a series $\sum u_n$, $B_n \to l \; (-\infty \leq l \leq +\infty)$ implies $A_n \to l$, then we can easily prove the following theorem, similar to Theorem 2.

Theorem II. *The propositions*

P': *if $B_n \geq 0$ for all n, then $A_n \geq 0$ for all n, and*

Q': *if $B_n = O_L(n^r)$ for a given $r > 0$, then $A_n = O_L(n^r)$,*

are equivalent; so are also the corresponding converse propositions.

Again, the propositions P' and Q' are true but not their converses.

In the proofs of Theorems I and II we shall have now to use the regular transformations

$$A^{\alpha}_{n} = \sum_{k=0}^{n}\binom{n}{k}\Delta^{n-k}\left(\frac{a_k}{b_k}\right)\cdot B^{\alpha}_{k},$$

and

$$B^{\alpha}_{n} = \sum_{k=0}^{n}\binom{n}{k}\Delta^{n-k}\left(\frac{b_k}{a_k}\right)\cdot A^{\alpha}_{k}.$$

Also we need the following two important properties of totally regular Hausdorff methods (See Basu (1), Lemmas 1, 2).

(*i*) A regular Hausdorff method A is *totally regular* (*i e.* $s_n \to l \; (-\infty \leq l \leq +\infty)$ implies $A_n \to l$) if and only if the corresponding factor sequence $\{a_n\}$ is totally monotone.

(*ii*) Of two Hausdorff methods with non-vanishing moment sequences each cannot totally include the other.

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NOTE ON THE COMMON TANGENTS OF A CUBIC AND ONE OF ITS SEXTACTIC CONICS

By

Haridas Bagchi

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Introduction. The main theme of this short paper is a general discussion on the common tangents of a bicursal cubic and one of its twentyseven sextactic conics. The first article treats of a general method (based on the use of *homogeneous* or *projective* coordinates) for finding the common tangents of a bicursal cubic Γ and an arbitrarily assigned conic (which may or may not be related to Γ). Then the second article concerns itself specifically with the common tangents of a (bicursal) cubic and a sextactic conic.

The results arrived at are believed to be mostly *new*.

1. Adopting *homogeneous* (or *projective*) coordinates, we may represent a bicursal (*i.e.*, non-singular) cubic Γ in the canonical form:

$$x^3+y^3+z^3 = 6mxyz. \tag{I}$$

Plainly then the polar line of an *arbitrary* point $P(x', y', z')$ w.r.t. Γ may be written as

$$x(x'^2+2my'z')+y(y'^2+2mz'x')+z(z'^2+2mx'y') = 0.$$

If this line be restricted to touch a given conic (say, II), whose tangential equation is

$$a\lambda^2+b\mu^2+c\nu^2+2f\mu\nu+2g\nu\lambda+2h\lambda\mu = 0, \tag{II}$$

the appropriate condition to be fulfilled has to be deduced from (II) by replacing λ, μ, ν respectively by

$$x'^2+2my'z', \quad y'^2+2mz'x', \quad z'^2+2mx'y'.$$

Consequently the locus of a *variable* point P, whose polar line touches II is the quartic Ω, given by the point-equation:

$$\begin{aligned} a(x^2+2myz)^2+b(y^2+2mzx)^2+c(z^2+2mxy)^2+2f(y^2+2mzx)(z^2+2mxy) \\ +2g(z^2+2mxy)(x^2+2myz)+2h(x^2+2myz)(y^2+2mzx) = 0. \end{aligned} \tag{III}$$

It is crystal-clear that this quartic Ω will cut the original cubic Γ at 12 points (real or imaginary), the tangent at each of which touches the given conic II. As a matter of fact, the tangents of Γ at its twelve points of intersection with Ω exhaust all the twelve tangents, that Γ has in common with II. We thus arrive at the following theorem.

Theorem A. *The twelve points of contact (real or imaginary) of a given bicursal* *cubic* (Γ) *with the twelve tangents, which it has in common with a given conic, can be designated as the (twelve) intersections of* (Γ) *with a certain quartic* (Ω)*.

If we now appeal to the familiar lemma that, whenever six out of the twelve points of intersection of a cubic and a quartic lie on a conic, the remaining six will also lie on a conic, we can immediately derive several corollaries (from Theorem A), *viz.*,

Corollary I.† *If a conic* Π *touches a bicursal cubic* Γ *at three points* A, B, C, *the six ordinary points on* Γ *(excluding* A, B, C*), the tangents at which touch* Π, *will lie on another conic.*

Corollary II. *If a conic* Π *has four-pointic and two-pointic contacts with a bicursal cubic* Γ *at two points* A *and* B *respectively, the six ordinary points on* Γ *(excluding* A, B*), the tangents at which touch* Π, *will lie on another conic.*

Corollary III. *If a conic* Π *has three-pointic contacts with a bicursal cubic* Γ *at two points* A, B, *the six ordinary points on* Γ *(excluding* A, B*), the tangents at which touch* Π, *will lie on another conic.*

Corollary IV. *The six ordinary points on a bicursal cubic, the tangents at which touch one of its sextactic conics, will lie on a certain conic.*

The next article will be devoted to a rather detailed discussion of Corollary IV, necessitating the adoption of the Cartesian mode of analysis from the very beginning.

2. We know that, if $S = 0$ be the osculating conic and $\xi = 0$ be the tangent to a given (bicursal) cubic Γ at one of its sextactic points (say, P), the equation to Γ must be amenable to the form:

$$S.\eta = \xi^3, \tag{I}$$

where $\eta = 0$ is a certain determinate line. As a matter of fact, this line ($\eta = 0$) is easily seen to be the tangent to Γ at that particular point of inflexion (say, O), which is the tangential of P.

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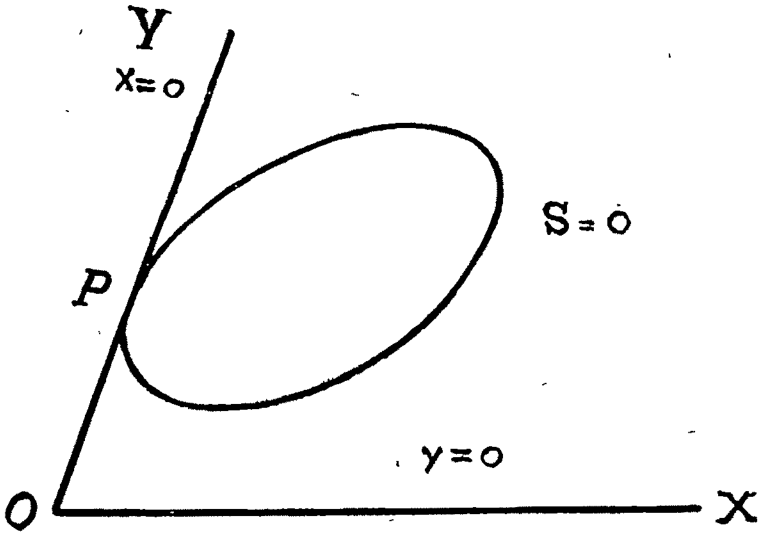
* A few words of explanation are needed to appreciate the full implication of Theorem A. *Under ordinary circumstances*, a proper (*i e.*, non-degenerate) quartic is defined uniquely, when it is conditioned to pass through *fourteen* arbitrarily assigned points, so that through *twelve* assigned points, there can, *in general*, be described a family of ∞^2 quartics. But in the *exceptional* case, when the twelve points in question happen to lie on one and the same cubic, no *proper* quartic will, *in general*, pass through them; in point of fact, the cubic itself, coupled with a perfectly *arbitrary* line in the plane, will ordinarily behave like a *degenerate* quartic, satisfying all the requirements

But then there are rare cases (as seen in Art. 1) when through twelve points on a cubic (say, $U = 0$), a proper (or non-degenerate) quartic can be drawn. It is remarkable that the existence of *one* such quartic (say, $V = 0$) implies the co-existence of an infinitude of other quartics of this description. For, if $L = 0$, denotes an *arbitrary* line (unrelated to $U = 0$ or $V = 0$), any quartic of the type ($V = UL$) will conform to all the requirements, *i.e.*, will pass through the twelve common points of ($U = 0$ and $V = 0$). Hence we can assert that the number of (non-degenerate) quartics that can be drawn through twelve points situated on cubic, is either *zero* or *infinity*. An illustration of this latter contingency is provided for in Theorem A.

† Mr. R. A. Roberts (1882) has proved Cor. I as an *independent* proposition in his book.



If, as shewn in the annexed figure, the said point of inflexion O be taken as the origin of Cartesian axes, and the lines ($\xi = 0$; $\eta = 0$) be taken respectively as the axes ($x = 0$, $y = 0$),—so that the axes are, in general, oblique—the equation (I) becomes

$$S.y - x^3 = 0, \tag{II}$$

it being implied that the conic given by

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \quad (A \equiv bc - f^2 = 0),$$

is the osculating (*i e.*, six-pointic) conic of Γ at the (sextactic) point P. If we now set

$$\left.\begin{aligned} u &= ax + hy + g, \\ v &= hx + by + f, \\ w &= gx + fy + c, \end{aligned}\right\} \tag{1}$$

we have the reciprocal relations:

$$\left.\begin{aligned} \Delta x &= Au + Hv + Gw, \\ \Delta y &= Hu + Bv + Fw, \\ \Delta &= Gu + Fv + Cw, \end{aligned}\right\} \tag{2}$$

provided that Δ stands for the *non-vanishing* determinant

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

and (A, B, C, F, G, H) are its co-factors.

Manifestly, then the polar line of a point (x, y), *w. r. t.* the cubic Γ, given by (II) has for its Cartesian equation:

$$X(2uy - 3x^2) + Y(2vy + S) + 2wy = 0,$$

where X, Y are the current (*i.e.*, running) coordinates.

Hence, bearing in mind that the tangential equation of the sextactic conic S is

$$A\lambda^2+B\mu^2+C\nu^2+2F\mu\nu+2G\nu\lambda+2H\lambda\mu=0, \qquad \text{(IV)}$$

(wherein the first term with a zero-coefficient *viz.*, A has been retained for the sake of symmetry), we readily realise that the twelve points on Γ, the tangents at which touch S as well, are the intersections of Γ with the quartic Ω, whose Cartesian equation is to be deduced from (IV) by substituting

$$2uy-3x^2, \quad 2vy+S, \quad 2wy$$

for λ, μ, ν respectively.

When these substitutions are actually effected, the L. S of (IV) becomes equal to

$$\begin{aligned} &A(2uy-3x^2)^2+B(2vy+S)^2+C4w^2y^2+4Fwy(2vy+S)\\ &\qquad\qquad+4Gwy(2uy-3x^2)+2H(2uy-3x^2)(2vy+S)\\ &=4y^2(Au^2+Bv^2+Cw^2+2Fvw+2Gwu+2Huv)+9Ax^4\\ &\qquad-12x^2y(Au+Hv+Gw)+4yS(Hu+Bv+Fw)+BS^2-6Hx^2S. \end{aligned}$$

Now

$$\begin{aligned} &Au^2+Bv^2+Cw^2+2Fvw+2Gwu+2Huv\\ &=u(Au+Hv+Gw)+v(Hu+Bv+Fw)+w(Gu+Fv+Cw)\\ &=\Delta(xu+yv+w), \text{ by (2)}\\ &=\Delta S, \text{ by (1)}. \end{aligned}$$

Hence, attending to the relation $A=0$, we can exhibit the Cartesian equation of the afore-mentioned quartic Ω in the form:

$$4y^2.\Delta S-12x^2y.\Delta x+4yS.\Delta y+BS^2-6Hx^2S=0,$$

i.e.

$$BS^2+(8\Delta y^2-6Hx^2)S-12\Delta x^3y=0. \qquad \text{(V)}$$

Regard being now had to the fact that the line $x=0$ touches the conic $S=0$ at the sextactic point P, it follows at once from the algebraic structure of the equation (V) that the quartic Ω must also have P for a sextactic point and $S=0$ for the osculating conic thereat. This result readily admits of verification by geometrical reasoning. For, S being the osculating conic at the *sextactic* point P on Γ, it follows that *six* adjacent points of Γ (and therefore also of S), lying in the immediate vicinity of P, possess the distinctive property that the tangents, drawn thereat to Γ, touch S as well. Consequently, the *six* adjacent points, just mentioned, must lie on the quartic Ω. This explains geometrically why P is a sextactic point and S is the corresponding six-pointic conic for the curve Ω.

Reverting to the equation (V) and combining it *algebraically* with the equation (II), we deduce:

$$S(BS-6Hx^2-4\Delta y^2)=0.$$

The obvious geometrical significance is that the twelve points of intersection of Ω with Γ lie, six by six, on the two conics

$$S = 0$$

and

$$BS - 6Hx^2 - 4\Delta y^2 = 0. \tag{VI}$$

This is obviously an analytical confirmation of Corollary IV of Art. 1. Combination of (II) and (VI) leads to

$$Bx^3 - 6Hx^2y - 4\Delta y^3 = 0 \tag{VII}$$

The palpable geometrical interpretation is that the six points of intersection of the conic (VI) with the cubic (II) lie, two by two, on the three right lines, defined by (VII).

The results may then be finalised as follows:

If S be the osculating conic of a bicursal cubic Γ at a sextactic point P, the six points on Γ—excluding P—the tangents at which touch S as well, lie on another conic, and are, besides, situated, two by two, on three right lines, that pass through that particular point of inflexion, which is the tangential of P.

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ON THE SOLUTION OF A PROBLEM ON INDETERMINATE EQUATIONS OCCURRING IN THE SULVA-SUTRAS

By

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In connection with the construction of sacrificial altars of specified sizes and shapes, a number of interesting indeterminate equations in several variables arise. Thus, in one particular form known as the square *Gārhapatya Vedi*, a side of the square is 1 *vyayama* or *purusha* (= 96 inches). The *Vedi* is to be constructed with 5 layers of bricks, each layer consisting of 21 bricks. The bricks may be rectangular or square, and of any size. The rifts of bricks in two consecutive layers must not coincide. Assuming the bricks to be square, a solution of the problem consists in using three kinds of square bricks, with $\frac{1}{6}$, $\frac{1}{4}$ and $\frac{1}{3}$ of a *vyayama* as a side. The first layer should be constructed with 9 bricks of the first kind and 12 bricks of the second kind, the second layer should contain 5 bricks of the third kind and 16 of the first. The third and fifth layers are replica of the first, while the fourth is a replica of the second. The solution depends on that of the simultaneous equations

$$\frac{x}{m^2}+\frac{y}{n^2}=1; \quad x+y=21 \tag{2}$$

There are only two solutions (in positive integers)

$$x=9, \quad y=12, \quad m=6, \quad n=4$$
$$x=5, \quad y=16, \quad m=3, \quad n=4.$$

A much more difficult problem arises in connection with the construction of the falcon-shaped fire altar (*Garudachayana shiti*). Its area is given to be 7½ square purushas, the altar must be constructed in 5 layers, and the number of bricks in any layer must be 200. As before, the rifts of bricks in any two successive layers must not coincide. Bōdhāyana uses 4 kinds of square bricks, whose number and dimensions depend upon the solution of the equations

$$\frac{x}{m}+\frac{y}{n}+\frac{z}{p}+\frac{u}{q}=7\tfrac{1}{2}; \quad x+y+z+u=200 \tag{2}$$

where m, n, p, q are squares of integers.

Apastamba uses 5 kinds of square bricks, thus making the problem depend upon the equations

$$\left.\begin{aligned} \frac{x}{m}+\frac{y}{n}+\frac{z}{p}+\frac{u}{q}+\frac{v}{r} &= 7\tfrac{1}{2} \\ x+y+z+u+v &= 200 \end{aligned}\right\} \qquad (3)$$

where m, n, p, q, r are squares of integers.

The above brief account is taken from Prof. Bibhuti Bhusan Datta's book (1932), "*The Science of the Sulba.*" The ancient *Sulvà-Sutras* which form what may be called annexures to the religious scriptures known as the *Śrauta Sutras* do not contain any mathematical exposition of the solutions of the mathematical problems involved in them, but give brief statements only about the solutions. Apastamba's statement with regard to (3) interpreted differently by different commentators leads to the following solutions:

$$\left.\begin{aligned} &\text{(i)} \quad x = 67,\ y = 58,\ z = 48,\ u = 18,\ v = 9,\ m = 16,\ n = 25,\ p = 36,\ q = 64,\ r = 100 \\ &\text{(ii)} \quad x = 12,\ y = 157,\ z = 9,\ u = 0,\ v = 22,\ m = 16,\ n = 25,\ p = 36,\ q = 64,\ r = 100 \end{aligned}\right\}$$

(Karavinda Swami)

(iii) $x = 10,\ y = 159,\ z = 9,\ u = 8,\ v = 14,\ m = 16,\ n = 25,\ p = 36,\ q = 64,\ r = 100$

(Kapardi Swami)

(iv)

$$\begin{aligned} x, y, z, u, v &= 70, 45, 9, 56, 20 \\ m, n, p, q, r &= 16, 25, 36, 64, 100 \\ x, y, z, u, v &= 74, 45, 52, 20, 9 \\ & 77, 42, 40, 32, 9 \\ m, n, p, q, r &= 16, 25, 64, 100, 144 \end{aligned}$$

(Sundararajà)

It is an interesting problem to discuss the method underlying the above particular solutions of (3). The method, at least in an empirical or incomplete form, must certainly have been known to the ancient Indians, and must necessarily be simple enough Prof. Datta makes no attempt to discuss the process underlying these solutions, though he explains fully the solutions of the very much simpler equations (1). It is the object of this note to discuss fully the solutions of the equations (3). Similar methods can be employed with respect to equations (2), and other problems of this type that are to be found in the *Sulva-Sutras.*

Multiplying the first of the equations in (1) by r and subtracting from the second, it follows that if r is the smallest of the squares m, n, p, q, r

$$7\tfrac{1}{2}r - 200 < 0 \qquad (4)$$

i.e. r must be one of the numbers 25, 16, 9, 4, 1. Similarly the largest of the squares m, n, p, q, r must be $> 200/7\frac{1}{2}$ *i.e.* ≥ 36

Hence no solution is possible taking $m, n, p, q, r = 25, 16, 9, 4, 1$.

Let us consider $m, n, p, q, r = 1, 25, 4, 9, 36$.

Put $x/m = k_1,\ y/m = k_2$ we get

$$9z+4u+v = 270-36(k_1+k_2)$$

$$z+u+v = 200-k_1-25k_2$$

Hence

$$8z+3u = 70-35k_1-11k_2$$

k_1+25k_2 and $36\ (k_1+k_2)$ are necessarily integers. k_1 is itself an integer since $m = 1$. It easily follows that k_2 is also an integer

Taking $k_1 = 1$, we can have $k_2 = 1$, 2 or 3, but $k_2 = 2$ or 3 lead to $8z+3u = 13$ or $8z+3u = 2$ which have no solutions. Hence $k_1 = k_2 = 1$ and we get the solution

$$\begin{array}{l} x, y, z, u, v = 1, 25, 0, 8, 166 \\ m, n, p, q, r = 1, 25, 4, 9, 36 \end{array} \tag{5}$$

using a similar method, we obtain the following solutions

$$\begin{array}{rl} x, y, z, u, v;\ m, n, p, q, r = 1, 25, 3, 0, 171; & 1, 25, 4, 16, 36 \\ 1, 50, 1, 4, 144; & ,, \end{array} \tag{6}$$

$$\left.\begin{array}{rl} 1, 25, 3, 12, 159; & 1, 25, 9, 16, 36 \\ 1, 25, 8, 0, 166; & ,, \\ 1, 50, 1, 8, 140; & ,, \end{array}\right\} \tag{7}$$

$$\left.\begin{array}{rl} 1, 1, 0, 9, 189; & 1, 4, 16, 9, 36 \\ 1, 0, 4, 10, 185; & ,, \end{array}\right\} \tag{8}$$

A slight variation of the method has to be adopted when we take

$$m, n, p, q, r = 25, 4, 16, 9, 36.$$

Let

$$x/25 = k_1 = \text{an integer}$$

$$u/9+v/36 = k_2,$$

so that

$$4y+z = 120-16(k_1+k_2).$$

Giving different values to k_1, k_2 such that $16(k_1+k_2)$ is an integer ≤ 120, we easily solve for y, z and u, v. We write down a few particular cases:

$$k_1 = 2,\ k_2 = 4$$

$$x, y, z;\ u, v = 50, \lambda, 24-4\lambda, 6-\lambda, 120+4\lambda.\ (\lambda = 0, 1, \ldots, 6). \tag{9}$$

$$k_1 = 4,\ k_2 = 3$$

$$x, y, z, u, v = 100, \lambda, 4-4\lambda, 7-\lambda, 89+4\lambda,\ (\lambda = 0, 1). \tag{10}$$

$$k_1 = 1,\ k_2 = 6\tfrac{1}{2}$$

$$x, y, z, u, v = 25, \lambda, 4-4\lambda, 18-\lambda, 158+4\lambda,\ (\lambda = 0, 1). \tag{11}$$

A list of further solutions obtained by similar processes is tabled below.

m	n	p	q	r	x	y	z	u	v	
1,	36,	64,	16,	4	1,	36,	$(100+4\lambda)$,	$(63-5\lambda)$,	λ	$(\lambda=0, 1, \ldots, 12)$
1,	81,	64,	16,	4	1,	81,	$40+4\lambda$,	$(78-5\lambda)$,	λ	$(\lambda=0, 1, \ldots, 15)$
1,	144,	64,	16,	4	1,	144,	4λ,	$(44-5\lambda)$,	$(11+\lambda)$	$(\lambda=0, 1, \ldots, 8)$
25,	36,	64,	16,	4	25,	36,	$(68+4\lambda)$,	$(71-5\lambda)$,	λ	$(\lambda=0, 1, \ldots, 14)$
1,	64,	16,	36,	9	1,	4λ,	$(16-\lambda)$,	$(178-4\lambda)$,	$(5+\lambda)$	$(\lambda=0, 1, \ldots, 15)$
4,	64,	16,	36,	9	4,	$(64-4\lambda)$,	λ,	$(110+4\lambda)$,	$(22-\lambda)$	$(\lambda=0, 1, \ldots, 16)$
49,	81,	64,	16,	4	49,	81,	4λ,	$(64-5\lambda)$,	$(\lambda+6)$	$(\lambda=0, 1, \ldots, 12)$
25,	64,	16,	36,	9	25,	4λ,	$(16-\lambda)$,	$(146-4\lambda)$,	$(13+\lambda)$	$(\lambda=0, 1, \ldots, 15)$
49,	64,	16,	36,	9	49,	4λ,	$(16-\lambda)$,	$(114-4\lambda)$,	$(21+\lambda)$	$(\lambda=0, 1, \ldots, 16)$
100,	64,	16,	36,	9	100,	4λ,	$(16-\lambda)$,	$(46-4\lambda)$	$(38+\lambda)$	$(\lambda=0, 1, \ldots, 11)$
121,	64,	16,	36,	9	121,	4λ,	$(16-\lambda)$,	$(18-4\lambda)$,	$(45+\lambda)$	$(\lambda=0, 1, \ldots, 4)$
16,	25,	64,	100,	144	$(k+5\lambda+3)$,	$(62-k)$,	$(448-4k-32\lambda)$,	$(2+4k)$,	$(27\lambda-315)$.	

$$\begin{bmatrix} k = 0, 1, \ldots, 16 \\ \lambda = 12, 13, 14 \text{ for every } k \end{bmatrix}.$$

It is now evident that a very large number of solutions can be written down and that the solutions given by the ancient Rishis and their commentators are included in these and are probably suitably chosen with due regard to the practical problem of constructing the altars. From the mathematical point of view, one has to complete the above discussion by showing that *the total number of possible solutions is finite.*

Considering the equations

$$\frac{x_1}{m_1}+\frac{x_2}{m_2}+\ldots+\frac{x_n}{m_n}=p.$$

$$x_1+x_2+\cdots+x_n=N,$$

where the m's are squares of integers. None of the x's can be greater than N, hence the number of solutions for x's is finite.

Let m_1 be the smallest of the m's. Multiplying the first equation by m_1 and subtracting the second

$$m_1p-\mathrm{N}<0 \qquad i.e. \quad m_1<N/p.$$

Let $\dfrac{x_1}{m_1}+\dfrac{x_2}{m^2}=p_1$ where $p_1 \lessgtr p$ and $m_1m_2p_1=$ an integer, we have

$$m_2 = \frac{m_1 x_2}{p_1 m_1 - x_1}$$

so that p_1 is such that $p_1 m_1 - x_1$ is an integer which divide $m_1 x_2$; hence the number of possible solutions for m_2 is finite. We now have

$$\frac{x_3}{m_3} + \cdots + \frac{x_n}{m_n} = p - p_1$$

$$x_3 + \cdots + x_n = N - (x_1 + x_2)$$

The argument is repeated. Thus, the number of possible solutions for the m's must be finite.

I conclude by thanking Dr. C. N. Srinivasiengar for his guidance in preparing this paper.

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ON THE ZEROS OF A NON-DIFFERENTIABLE FUNCTION

By

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The function (Hobson, 1907) represented by $\sum_1^\infty a^n \sin b^n x$, $(a < 1)$ where b is an integer of the form $4p+1$, is continuous but nowhere has a finite differential coefficient if $ab \geq 1$ and one of the two conditions $ab > 1+3\pi/2$, or $ab^2 > 1+3\pi^2$.

The properties of the zeros of this function has not been considered before. Here I shall find the zeros of this function and find some prperties of the zeros.

1. Theorem 1. *The zeros of* $w(x) = \sum_1^\infty a^n \sin b^n x$ *between* $-\pi$ *and* π *are* $\pm(\pi/b^k)(1+\lambda_k/2)$ *where k is any positive integer,* $ab > 1+3\pi/2$ *and* $b > 1$.

Proof.

$$w(\pi/b^k) = a \sin(\pi/b^{k-1}) + a^2 \sin(\pi/b^{k-2}) + \cdots > 0$$

as each term of the above series is either > 0 or $= 0$.

$$\begin{aligned} w(3\pi/2b^k) &= \sum_1^\infty a^n \sin(3\pi/2b^{n-k}) \\ &= a \sin(3\pi/2b^{k-1}) + a^2 \sin(3\pi/2b^{k-2}) + \cdots + a^{k-1}\sin(3\pi/2b) - (a^k + a^{k+1} + \cdots \text{to infinity}) \\ &= -\left[\frac{a^k}{1-a} - a^{k-1}\sin(3\pi/2b) - a^{k-2}\sin(3\pi/2b^2) - \cdots - a\sin(3\pi/2b^{k-1})\right]. \end{aligned}$$

Now

$$\begin{aligned} a^{k-1}\sin(3\pi/2b) + a^{k-2}\sin(3\pi/2b^2) + \cdots + a\sin(3\pi/2b^{k-1}) \\ < (a^{k-1} + a^{k-2}/b + \cdots + 1/b^{k-1})(3\pi/2b) \\ < (3\pi/2b)a^{k-1}(1 + 1/(ab) + 1/(ab)^2 + \cdots) \\ < (3\pi/2b)a^{k-1}/(1-1/(ab) < (3\pi/2)a^k/(ab-1) < a^k < a^k(1-a) \end{aligned}$$

since $ab-1 > 3\pi/2$.

Hence $w(3\pi/2b^k) < 0$. Hence there must be a value of x between π/b^k and $3\pi/2b^k$ for which $w(x) = 0$. Hence the zeros of $w(x)$ between $-\pi$ and π will be $(\pi/b^k)(1+\lambda_k/2)$ where k is any integer and $0 < \lambda_k < 1$. It is evident that $\pm\pi$ will also be zeros of $w(x)$.

2. If ξ be a number such that $D^+w(\xi) = D^-w(\xi) = \infty$; $D_+w(\xi) = D_-w(\xi) = -\infty$ then ξ will be a knot point of $w(x)$.

If η be a number such that $D^+w(\eta) = D_+w(\eta) = \pm\infty$, $D^-w(\eta) = D_-w(\eta) = \mp\infty$ then η will be called a cusp.

Theorem 2. *If ξ is a zero of $w(x)$ and if ξ is a knot point, ξ is necessarily a limiting point of zeros of $w(x)$.*

Proof. If not let ξ be an isolated zero of $w(x)$. Then we can find a positive number k such that in the interval $(\xi-h, \xi+h)$, where $0 < h < k$, $w(x)$ has no zero other than ξ.

(*i*) If $w(\xi+h)$ and $w(\xi-h)$ are of the same sign, say positive, $w(\xi+h)/h$ is positive and $w(\xi-h)/(-h)$ is negative and $D_+w(\xi) \geq 0$, so that ξ is not a knot point, which contradicts the hypothesis.

(*ii*) It may be $w(\xi+h)$ and $w(\xi-h)$ are of opposite sign, say $w(\xi+h) > 0$ and $w(\xi-h) < 0$. Then $w(\xi+h)/h$ and $w(\xi-h)/(-h)$ are both positive. Hence the relations $D_+w(\xi) = D_-w(\xi) = -\infty$ are impossible, so that ξ is not a knot point, which contradicts the hypothesis, Hence we have the theorem.

Theorem 3. *The zeros of $w(x)$, other than those that may happen to be cusps, are limiting points of zeros.*

Proof. (*i*) If $w(\xi+h)$ and $w(\xi-h)$ have the same sign $\lim\limits_{h\to+0} w(\xi+h)/h$ can not have the upper limit $+\infty$ and lower limit $-\infty$. Similarly, $\lim\limits_{h\to+0} w(\xi-h)/(-h)$ can not have $+\infty$ and $-\infty$ as upper and lower limits.

(*ii*) If $w(\xi+h)$ and $w(\xi-h)$ are of opposite signs for all h, $0 < h < k$ then neither $w(\xi+h)/h$ nor $w(\xi-h)/(-h)$ can have $+\infty$ and $-\infty$ as upper and lower limits as $h \to +0$. Hence we have the theorem.

Similar properties have been proved by Srinivasienger (1938) in case of Weierstrass' non-differentiable function.

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VERSCHIEDENE ZAHLENTHEORETISCHE UNTERSUCHUNGEN UND DIOPHANTISCHE PROBLEME

By

Alfred Moessner

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I. (1) Besteht die Relation $a+b+c+d=0$, dann ist

$$a^3+b^3+c^3+d^3 = 3(bcd+cda+dab+abc);$$

ist $a^3+b^3+c^3+d^3=0$, dann ist

$$a^9+b^9+c^9+d^9 = 3[(bcd)^3+(cda)^3+(dab)^3+(abc)^3].$$

Exempel: $a=3$, $b=4$, $c=5$, $d=6$ ergibt $3^3+4^3+5^3-6^3=0$, also

$$3^9+4^9+5^9-6^9 = 3[(-120)^3+(-90)^3+(-72)^3+60^3].$$

(2) (α) Besteht $a+b+c=d+e+f$ nur aus Primzahlen, dann gibt

$$\sqrt{(8a+1)},\ \sqrt{(8b+1)},\ \sqrt{(8c+1)} \overset{2}{=} \sqrt{(8d+1)},\ \sqrt{(8e+1)},\ \sqrt{(8f+1)}.$$

Exempel: $a=1$, $b=15$, $c=21$, $d=3$, $e=6$, $f=28$ (a, b, c, d, e, f sind Dreieckszahlen) ergibt

$$\sqrt{(8.1+1)},\ \sqrt{(8.15+1)},\ \sqrt{(8.21+1)} \overset{2}{=} \sqrt{(8.3+1)},\ \sqrt{(8.6+1)},\ \sqrt{(8.28+1)},$$

also $3, 11, 13 \overset{2}{=} 5, 7, 15$.

(β) Ist $A_1, A_2, \ldots, A_5 \overset{2}{=} B_1, B_2, \ldots, B_5$, wo sämtliche A_i und B_i sind Dreieckszahlen, dann ist auch

$$\sqrt{(8A_1+1)},\ \sqrt{(8A_2+1)}, \ldots, \sqrt{(8A_5+1)} \overset{4}{=} \sqrt{(8B_1+1)},\ \sqrt{(8B_2+1)} \ldots, \sqrt{(8B_5+1)}.$$

Exempel: $A_1=1$, $A_2=15$, $A_3=45$, $A_4=153$, $A_5=171$; $B_1=3$, $B_2=6$, $B_3=66$, $B_4=120$, $B_5=190$, ergibt

$$3, 11, 19, 35, 37 \overset{4}{=} 5, 7, 23, 31, 39.$$

II. Wir lösen auf verschiedene Arten die Identität

$$\left.\begin{array}{l} A+B+B = C+D+D = X^2 \\ A\,.\,B\,.\,B = C\,.\,D\,.\,D = Y^2 \end{array}\right\}$$

(1) Wir wählen für A und C je eine beliebige rationale Quadratzahl, jedoch $A \neq C$. Setzt man nun $\frac{1}{2}(\sqrt{A}+\sqrt{C}) = g$, dann ist $B = g.\sqrt{C}$, $D = g.\sqrt{A}$.

Exempel (α): $A=49$, $C=81$, $g=\frac{1}{2}(\sqrt{49}+\sqrt{81})=8$; $B=8.\sqrt{81}=72$, $D=8.\sqrt{49}=56$. Wir bekommen also die Lösung

$$\left.\begin{array}{l} 49+72+72 = 81+56+56 = (\sqrt{193})^2 \\ 49\,.\,72\,.\,72 = 81\,.\,56\,.\,56 = 504^2 \end{array}\right\}.$$

Exempel (β): $A = 3^4$, $C = 5^4$, $g = \frac{1}{2}(\sqrt{3^4}+\sqrt{5^4}) = 17$ ergibt die Lösung

$$\left.\begin{array}{l} 3^4+425+425 = 5^4+153+153 = (\sqrt{931})^2 \\ 3^4 \,.\, 425 \,.\, 525 = 5^4 \,.\, 153 \,.\, 153 = 3825^2 \end{array}\right\}.$$

Man erkennt sofort, dass A und C beliebige gerade Potenzen sein könen.

(2) Soll nun X eine rationale Zahl sein, dann müssen $\sqrt{A}$ und $\sqrt{C}$ der Kondition genügen $\sqrt{A}$, $\sqrt{C}$, $(\sqrt{A}+\sqrt{C}) \overset{n}{=} E, E$, $(n = 2, 4)$, wobei $\sqrt{A}$, $\sqrt{C}$ und E ganze rationale Zahlen sind. Diese Kondition ist erfüllt, wenn man setzt $\sqrt{A} = u^2-v^2$, $\sqrt{C} = 2uv+v^2$, $E = u^2+uv+v^2$, wobei $v < u$ und v und u rational sind.

Exempel: $u = 2$, $v = 1$ ergibt $\sqrt{A} = 3$, $\sqrt{C} = 5$, also $A = 9$, $C = 25$, $g = \frac{1}{2}(3+5) = 4$, $D = 4.3 = 12$, $B = 4.5 = 20$, also die Lösung

$$\left.\begin{array}{l} 9+20+20 = 25+12+12 = 7^2 \\ 9 \,.\, 20 \,.\, 20 = 25 \,.\, 12 \,.\, 12 = 60^2 \end{array}\right\}.$$

(3) Setzt man $B = 2uv+v^2$, $D = u^2-v^2$, wobei $u > v$ und u und v ganze rationale Zahlen sind, dann gilt

$$\left.\begin{array}{l} A+B+B = C+D+D = X^2 \\ A \,.\, B \,.\, B = C \,.\, D \,.\, D = Y^2 \end{array}\right\},$$

wenn $A = 2D^2/(B+D)$ und $C = 9B^2/(B+D)$.

Exempel: $u = 1.$ ergibt $B = 5$, $D = 3$, $A = 18/8 = 9/4$, $C = 25/4$. Diese numerische Lösung ist identisch mit dem Exempel unter (2).

(4) Es gilt allgemein

$$\left.\begin{array}{l} (2F^2)+(FG+G^2)+(FG+G^2) = (2G^2)+(FG+F^2)+(FG+F^2) = 2H^2 \\ (2F^2) \,.\, (FG+G^2) \,.\, (FG+G^2) = (2G^2) \,.\, (FG+F^2) \,.\, (FG+F^2) = 2(F^2G+FG^2)^2 \end{array}\right\}$$

in rationalen Zahlen, wenn man setzt $F = u^2-v^2$, $G = 2uv+v^2$, wo $u > v$ und u und v rationale Zahlen sind.

(5) Ist $F = u^2-v^2$ und $G = 2uv+v^2$, wo $u > v$ und u und v rational, $d = \frac{1}{2}(G-F)$, dann gilt die Relation

$$\left.\begin{array}{l} F^2+d.G+d.G = G^2 - d.G - d.G \\ F^2 \,.\, d.G \,.\, d.G = G^2.-d.G.-d.G = (F.dG)^2 \end{array}\right\}.$$

Exempel: $u = 4$, $v = 3$ ergibt $F = 7$, $G = 33$, $d = \frac{1}{2}(33-7) = 13$ und wir bekommen das Resultat

$$\left.\begin{array}{l} 49+429+429 = 1089 - 91 - 91 = 907 \\ 49 \,.\, 429 \,.\, 429 = 1089.-91,-91 = (7.429)^2 \end{array}\right\}.$$

(6) Besteht die Relation

$$\left.\begin{aligned} M_1+M_2+M_3 &= P_1+P_2+P_3 \\ M_1 \,.\, M_2 \,.\, M_3 &= P_1 \,.\, P_2 \,.\, P_3 \end{aligned}\right\}$$

und ist $s = M_1+M_2+M_3$, dann gilt auch die Identität

$$s-2M_1,\ s-2M_2,\ s-2M_3 \overset{n}{=} s-2P_1,\ s-2P_2,\ s-2P_3 \quad (d = 1, 3).$$

Wenden wir dieses Theorem auf die Relation

$$\left.\begin{aligned} A+B+B &= C+D+D = X^2 \\ A \,.\, B \,.\, B &= C \,.\, D \,.\, D = Y^2 \end{aligned}\right\}$$

[*cf.* (3)!] oder auf die Relation

$$\left.\begin{aligned} F^2+d.G+d.G &= G^2 \ -d.G \ -d.G \\ F^2 \,.\, d.G \,.\, d.G &= G^2.-d.G.-d.G \end{aligned}\right\}$$

[*cf.* (5)!] an, dann bekommen wir ganzzahlige rationale Lösungen zu der Identität

$$\left.\begin{aligned} K \ +R^2+R^2 &= T \ +W^2+W^2 \\ K^3+R^6+R^6 &= T^3+W^6+W^6 \end{aligned}\right\}.$$

Exempel: Aus 9, 20, 20 $\overset{+}{\div}$ 25, 12, 12*, wo $\sum = 7^2$, $\prod = 60^2$, folgt, da $s = 9+20+20 = 49$, die Identität $(49-18), (49-40), (49-40) \overset{n}{=} (49-50), (49-24), (49-24)$ für $n = 1, 3$, also

$$\left.\begin{aligned} 31 \ +3^2+3^2 &= (-1) \ +5^2+5^2 \\ 31^3+3^6+3^6 &= (-1)^3+5^6+5^6 \end{aligned}\right\}.$$

Die gleiche Identität folgt aus 9, 5, 5 $\overset{+}{\div}$ 25, (−3), (−3).

(7) Zur Identität

$$\left.\begin{aligned} K \ +R^2+R^2 &= T \ +W^2+W^2 \\ K^3+R^6+R^6 &= T^3+W^6+W^6 \end{aligned}\right\}$$

gelangen wir auch durch folgende Methode.

Wir setzen

$$s = (u^2+uv+v^2), \quad R = u^2-v^2, \quad K = s-2R^2, \quad W = 2uv+v^2 \quad T = s-2W^2,$$

wobei $u > v$.

Exempel: $u = 2$, $v = 1$ ergibt $s = 49$, $R = 3$, $K = 38$, $W = 5$, $T = 1$, also die Lösung unter (6).

(8) Im Hinblick auf vorstehende Darlegungen sei bemerkt, dass generell gilt

$$L_1+L_2+L_3+ \cdots L_r+\{(L_1+L_2+L_3+ \cdots +L_r)/(L_1 \,.\, L_2 \,.\, L_3 \,.\cdots .\, L_r-1)\}$$
$$= L_1 \,.\, L_2 \,.\, L_3 \,.\cdots .\, L_r \,.\, \{(L_1+L_2+L_3+ \cdots +L_r)/(L_1 \,.\, L_2 \,.\, L_3 \,.\cdots .\, L_r-1)\}$$

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* Das Symbol $\overset{+}{\div}$ bedeutet Summen-und Produktengleichheit.

III. (1) Ist $M_1+M_2+M_3+\cdots+M_r = N_1+N_2+N_3+\cdots+N_r$, aber $\sum_{i=1}^{r} M_i^2 \neq \sum_{i=1}^{r} N_i^2$ und $\sum_{i=1}^{r} M_i^3 \neq \sum_{i=1}^{r} N_i^3$ und ist

$$K = \left(\sum_{i=1}^{r} N_i^3 - \sum_{i=1}^{r} M_i^3\right) \Big/ \left\{3.\left(\sum_{i=1}^{r} N_i^2 - \sum_{i=1}^{r} M_i^2\right)\right\},$$

dann gilt $K-M_1, K-M_2, K-M_3, \ldots, K-M_r = K-N_1, K-N_2, K-N_3, \ldots, K-N_r$, für $n = 1, 3$. Hiebei kann $M_1 = M_2 = M_3 = \cdots = M_r$ sein. es ist also allgemein

$$r.L^n = P_1^n + P_2^n + \cdots + P_r^n$$

$(n = 1, 3)$ lösbar.

(2) Ist $r = 3$, so bekommen wir Lösungen zur Identität $L, L, L \overset{n}{=} P_1, P_2, P_3$ für $n = 1, 3$, wenn wir setzen

$$K = \left(\sum_{i=1}^{3} N_i^3 - 3.M_1^3\right) \Big/ \left\{3.\left(\sum_{i=1}^{3} N_i^2 - 3.M_1^2\right)\right\}.$$

Exempel: $M_1 = M_2 = M_3 = 3$, $N_1 = 1$, $N_2 = 2$, $N_3 = 6$ ergibt $K = \dfrac{(225-81)}{\{3\,(41-27)\}} = \dfrac{24}{7}$, also die Identität $3, 3, 3 \overset{n}{=} 17, 10, (-18)$ für $n = 1, 3$.

(3) Ist $a_1, a_2, a_3 \overset{n}{=} b_1, b_2, b_3$ für $n = 1, 3$ und setzt man $k = a_1+a_2+a_3$, dann gilt

$$\tfrac{1}{2}(k-a_1),\ \tfrac{1}{2}(k-a_2),\ \tfrac{1}{2}(k-a_3) \overset{+}{=} \tfrac{1}{2}(k-b_1),\ \tfrac{1}{2}(k-b_2),\ \tfrac{1}{2}(k-b_3).$$

Wenden wir dieses Theorem auf die Relation $L, L, L \overset{n}{=} P_1, P_2, P_3$ für $n = 1, 3$ an, so bekommen wir Lösungen zu $D, D, D \overset{+}{=} E_1, E_2, E_3$, wo die E_i Kubikzahlen sind.

Exempel: Aus $3, 3, 3 \overset{n}{=} 17, 10, (-18)$ für $n = 1, 3$ folgt $k = 9$, also

$$\tfrac{1}{2}(9-3),\ \tfrac{1}{2}(9-3),\ \tfrac{1}{2}(9-3) \overset{+}{=} \tfrac{1}{2}(9-17),\ \tfrac{1}{2}(9-10),\ \tfrac{1}{2}(9-(-18)),$$

also, auf ganze Zahlen gebracht,

$$\left.\begin{aligned} 6+6+6 &= (-8)+(-1)+27 \\ 6\,.\,6\,.\,6 &= (-8)\,.\,(-1)\,.\,27 \end{aligned}\right\}.$$

(4) (α) Besteht die Relation $M_1, M_2, M_3 \overset{+}{=} P_1, P_2, P_3$ [*cf.* II, (6) !] und ist $s = M_1+M_2+M_3$, dann gilt aus die Identität

$$M_1, M_2, M_3, s-P_1, s-P_2, s-P_3 \overset{4}{=} P_1, P_2, P_3, s-M_1, s-M_2, s-M_3.$$

Dies Theorem verwenden wir zur Lösung der Identität

$$\left.\begin{array}{llll} A,\ B,\ B,\ E_1,\ E_2,\ E_3,\ E_4,\ E_5 & \overset{5}{=} D,\ D,\ C,\ F_1,\ F_2,\ F_3,\ F_4,\ F_5, & & \\ A+B+B & = D+D+C & & = X^2 \\ A\,.\,B\,.\,B & = D\,.\,D\,.\,C & & = Y^2 \\ \qquad E_1+E_2+E_3+E_4+E_5 = & & F_1+F_2+E_3+F_4+F_5 & \end{array}\right\}$$

Wir gehen aus von der Relation $A+B+B \overset{+}{\underset{\cdot}{=}} D+D+C$, wo $A+B+B = X^2$ und $A.B.B = Y^2$. Auf sie wenden wir das in diesem Absatze angegebene Theorem an und setzen $s = A+B+B$, dann gilt die Identität

$$A,\ B,\ B,\ s-D,\ s-D,\ s-C \overset{4}{=} D,\ D,\ C,\ s-A,\ s-B,\ s-B,$$

Ist

$$m_1,\ m_2,\ m_3,\ \ldots,\ m_r \overset{4}{=} p_1,\ p_2,\ p_3\ \ldots,\ p_r,$$

dann gilt für beliebiges h die Identität

$$\begin{aligned} &m_1,\ m_2,\ m_3,\ \ldots,\ m_r,\ p_1+h,\ p_2+h,\ \ldots,\ p_r+h \\ &\qquad \overset{5}{=} p_1,\ p_2,\ p_3,\ \ldots,\ p_r,\ m_1+h,\ m_2+h,\ m_3+h,\ \ldots,\ m_r+h. \end{aligned}$$

Wir wenden dies Theorem an auf die Identität

$$A,\ B,\ B,\ s-D,\ s-D,\ s-C \overset{4}{=} D,\ D,\ C,\ s-A,\ s-B,\ s-B$$

und setzen $d = s-D-B$ und bekommen

$$\begin{aligned} &A,\ B,\ B,\ s-D,\ s-D,\ s-C,\ D+s-D-B,\ D+s-D-B,\ C+s-D-B, \\ &\qquad s-A+s-D-B,\ s-B+s-D-B,\ s-B+s-D-B \\ &\overset{5}{=} D,\ D,\ C,\ s-A,\ s-B,\ s-B,\ A+s-D-B,\ B+s-D-B,\ B+s-D-B \\ &\qquad s-D+s-D-B,\ s-D+s-D-B,\ s-C+s-D-B, \end{aligned}$$

also:

$$\begin{aligned} &A,\ A,\ B,\ \underline{s-D},\ \underline{s-D},\ s-C,\ \underline{s-B},\ \underline{s-B},\ C+s-D-B,\ 2s-A-D-B, \\ &\qquad 2s-D-2B,\ 2s-D-2B \\ &\overset{5}{=} D,\ D,\ C,\ s-A,\ \underline{s-B},\ \underline{s-B},\ A+s-D-B,\ \underline{s-D},\ \underline{s-D},\ 2s-2D-B, \\ &\qquad 2s-2D-B,\ 2s-C-D-B. \end{aligned}$$

Da die unterstrichenen Glieder wegfallen, bekommen wir die Identität

$$\begin{aligned} &A,\ B,\ B,\ s-C,\ C+s-D-B,\ 2s-A-D-B,\ 2s-D-2B,\ 2s-D-2B \\ &\qquad \overset{5}{=} D,\ D,\ C,\ s-A,\ A+s-D-B,\ 2s-2D-B,\ 2s-2D-B,\ 2s-C-D-B. \end{aligned}$$

Exempel: $A = 9$, $B = 20$, $D = 12$, $C = 25$ [*cf.* Exempel unter II, (6)!] ergibt das Resultat

$$9,\ 20,\ 20,\ 24,\ 42,\ 57,\ 46,\ 46 \overset{5}{=} 12,\ 12,\ 25,\ 40,\ 26,\ 54,\ 54,\ 41,$$

wo

$$9,\ 20,\ 20 \overset{+}{\underset{\cdot}{=}} 12,\ 12,\ 25$$

und

$$24+42+57+46+46 = 40+26+54+54+41.$$

(β) Gilt die Relation

$$G_1,\ G_2\ \ldots,\ G_8 \overset{5}{=} H_1,\ H_2,\ \ldots,\ H_8,$$

wo $G_1 < G_2 < G_3 < G_4 \leq \ldots \leq G_8$ und $H_1 < H_2 < H_3 \leq \ldots \leq H_8$

ferner

$$G_1+G_8 = G_2+G_7 = G_3+G_6 = G_4+G_5 = H_1+H_8 = H_2+H_7 = H_3+H_6 = H_4+H_5,$$

dann gilt auch

$$G_8-G_1,\ G_7-G_2,\ G_6-G_3,\ G_5-G_4 \overset{n}{=} H_8-H_1,\ H_7-H_2,\ H_6-H_2,\ H_5-H_4$$

für $n = 2, 4$.

Wir wenden dieses Theorem auf das Exempel unter (α) an, nachdem wir dessen Zahlen nach der Grösse ordneten: Aus

$$9,\ 20,\ 20,\ 24,\ 42,\ 46,\ 46,\ 57 \overset{5}{=} 12,\ 12,\ 25,\ 26{,}40,\ 41,\ 54,\ 54$$

folgt also

$$57-9,\ 64-20,\ 46-20,\ 42-24 \overset{n}{=} 54-12,\ 54-12,\ 41-25,\ 40-26 \quad (n = 24),$$

also

$$48,\ 26,\ 26,\ 18 \overset{n}{=} 42,\ 42,\ 16,\ 14 \quad (n = 2,\ 4)$$

also

$$24,\ 13,\ 13,\ \ 9 \overset{n}{=} 21,\ 21,\ \ 8,\ \ 7 \quad (n = 2,\ 4).$$

Da sich aus der letzten Identität unter (α) immer eine Relation der Form $G_1, G_2, \ldots, G_8 \overset{5}{=} H_1, H_2, \ldots, H_8$ mit obigen Konditionen und mit der Kondition $G_2 = G_3$ und $G_6 = G_7$, ferner $H_1 = H_2$ und $H_7 = H_8$ bilden lässt, so können wir auch immer aus ihr eine Lösung zu $Q, R, R, S \overset{n}{=} U, U, V, W$ für $n = 2. 4$ bekommen wo für ich vorhin bereits ein Exempel gab.

(γ) Gilt $G_1, G_2, \ldots, G_8 \overset{5}{=} H_1, H_2, \ldots, H_8$ mit den Konditionen unter (β), dann gilt auch

$$G_1G_8+G_2G_7+G_3G_6+G_4G_5 = H_1H_8+H_2H_7+H_3H_6+H_4H_5.$$

Exempel: Aus dem Exempel unter (β) folgt

$$9.57+20.46+20.46+24.42 = 12.54+12.54+25.41+26.40 = 3361.$$

GUNZENHAUSEN (GERMANY—BAYERN)

ON THE TOTAL RELATIVE STRENGTH OF THE RIESZ AND HÖLDER METHODS

By

S. K. BASU

1. INTRODUCTION. It is well known that, for $\kappa \geq 0$, the (C, κ), (H, κ) and (R, n, κ) methods of summability are *equivalent* in the sense that if a series is summable to l[*] by any one of the three methods, then it is also summable to l by the other two (See Hausdorff, 1921, p. 90; Hobson, 1926, p. 90). The question then naturally arises as to how far this result remains true when l is infinite. The present paper is concerned with this problem of total relative strength[†] of the Riesz and Hölder methods, the corresponding problems for the Hölder and Cesàro, and Riesz and Cesàro methods having been disposed of by the author in two recent papers (Basu (1), (2)). A part of the present problem, however, remains incomplete and is a subject for future investigation.[‡]

In the last section of the present paper is given a table containing the results of this paper and of the two above quoted papers by the author.

2. The Hölder mean H_n^κ of order $\kappa > -1$ for a series $\sum_{\nu=0}^{\infty} a_\nu$ of *real* terms (written in short as $\sum a_n$) for which $s_n = \sum_{\nu=0}^{n} a_\nu$ is defined by (See Hausdorff, p. 83)

$$H_n^\kappa = \sum_{m=0}^{n} \binom{n}{m} \Delta^{n-m} \mu_m . s_m \qquad (n = 0, 1, 2, \ldots), \tag{1}$$

where

$$\mu_n = \frac{1}{(n+1)^\kappa};$$

whence

$$s_n = \sum_{m=0}^{n} \binom{n}{m} \Delta^{n-m} \left(\frac{1}{\mu_m}\right) H_m^\kappa. \tag{2}$$

Let $\sigma^\kappa(\omega)$ and $R^\kappa(\omega)$ denote respectively the Riesz sum and Riesz mean of order $\kappa \geq 0$ for the series $\sum a_n$; so that

$$\sigma^\kappa(\omega) = \sum_{r=0}^{[\omega]} (\omega - r)^\kappa a_r \qquad (\omega > 0),$$

and

$$R^\kappa(\omega) = \frac{\sigma^\kappa(\omega)}{\omega^\kappa}. \tag{3}$$

The following theorems will be proved.

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[*] Unless otherwise stated, l denotes a finite number

[†] A method of summability A is said to be *stronger* than another B, if every series summable B to l is also summable A to l (See Widder, 1946, p. 121). If the B summability of a series to l $(-\infty \leq l \leq +\infty)$ implies its A summability to l, then we say that A is *totally stronger* than B

[‡] See the last footnote in § 2 below.

Theorem 1. For $0 < \kappa < 1$, $H_n^\kappa \to +\infty$ does not imply $R^\kappa(\omega) \to +\infty$.*

Theorem 2. If $\kappa > 1$, then $H_n^\kappa \to +\infty$ does not imply $R^\kappa(\omega) \to +\infty$.

Theorem 3.† For $\kappa > 1$, $R^\kappa(\omega) \to +\infty$ does not imply $H_n^\kappa \to +\infty$.‡

3. *Proof of Theorem 1.* To prove Theorem 1, instead of the continuous Riesz mean, we preferably consider the discontinuous mean α_n^κ defined by the transformation (See Riesz, 1909, p. 19 or Agnew, 1933, p. 532)

$$\alpha_n^\kappa = \sum_{r=0}^{n-1}\left(1-\frac{r}{n}\right)^\kappa a_r \qquad (n = 1, 2, 3, \ldots). \tag{4}$$

In case $\alpha_n^\kappa \to l$, we say that the series Σa_n is summable (A, κ) to l. We note from (3) that

$$R^\kappa(n) = \sum_{r=0}^{n}\left(1-\frac{r}{n}\right)^\kappa a_r = \alpha_n^\kappa,$$

so that as $R^\kappa(\omega) \to +\infty$, $\alpha_n^\kappa \to +\infty$.

It is known (Riesz, 1924, p. 418 or Agnew, 1933, p. 533) that, for $0 < \kappa < 1$,

$$(C, \kappa) \approx (A, \kappa),$$

i.e., the two methods of summability are equivalent; and since, for $\kappa > -1$,

$$(H, \kappa) \approx (C, \kappa),$$

it follows that, for $0 < \kappa < 1$,

$$(H, \kappa) \approx (A, \kappa). \tag{5}$$

From (4) by Abel's formula on partial summation,

$$\alpha_n^\kappa = \sum_{r=0}^{n-1}\left\{\left(1-\frac{r}{n}\right)^\kappa - \left(1-\frac{r+1}{n}\right)^\kappa\right\} s_r;$$

so that by (2),

$$\begin{aligned}\alpha_n^\kappa &= \sum_{r=0}^{n-1}\left\{\left(1-\frac{r}{n}\right)^\kappa - \left(1-\frac{r+1}{n}\right)^\kappa\right\}\sum_{\nu=0}^{r}\binom{r}{\nu}\Delta^{r-\nu}(\nu+1)^\kappa H_\nu^\kappa\\ &= \sum_{\nu=0}^{n-1} H_\nu^\kappa \sum_{r=\nu}^{n-1}\binom{r}{\nu}\left\{\left(1-\frac{r}{n}\right)^\kappa - \left(1-\frac{r+1}{n}\right)^\kappa\right\}\Delta^{r-\nu}(\nu+1)^\kappa,\end{aligned}$$

or

$$\alpha_n^\kappa = \sum_{\nu=0}^{n-1} a_{n\nu} H_\nu^\kappa, \tag{6}$$

---

* In the trivial cases $\kappa = 0, 1$, $H_n^\kappa \to +\infty$ implies $R^\kappa(\omega) \to +\infty$ and conversely. For, when $\kappa = 0$ or 1, the Cesàro mean of order κ, $C_n^\kappa \to +\infty$ implies $R^\kappa(\omega) \to +\infty$ (Basu (2), Theor 1) Also, H_n^κ and C_n^κ are indentical in these cases, so that $H_n^\kappa \to +\infty$ implies $R^\kappa(\omega) \to +\infty$. Again, for $\kappa = 0, 1$, $R^\kappa(\omega) \to +\infty$ implies $C_n^\kappa \to +\infty$ (Basu (2), Theor. 3) and therefore $H_n^\kappa \to +\infty$.

† Theorem 4 (Basu (2)) is included in this theorem. For, if $R^\kappa(\omega) \to +\infty$ implies $C_n^\kappa \to +\infty$, then it will follow with the aid of Theorem 3 (Basu (1)) that, for $\kappa > 1$, $R^\kappa(\omega) \to +\infty$ implies $H_n^\kappa \to +\infty$.

‡ The question whether or not $R^\kappa(\omega) \to +\infty$ implies $H_n^\kappa \to +\infty$, for $0 < \kappa < 1$, is yet to be cosidered.

where

$$a_{n\nu}=\sum_{r=\nu}^{n-1}\binom{r}{\nu}\left\{\left(1-\frac{r}{n}\right)^{\kappa}-\left(1-\frac{r+1}{n}\right)^{\kappa}\right\}\Delta^{r-\nu}(\nu+1)^{\kappa}. \tag{7}$$

Then, since in virtue of (2), $\{H_n^{\kappa}\}$ can be any sequence, it follows from (5) that the transformation (6) is regular.* Hence $H_n^{\kappa}\to+\infty$ will imply $\alpha_n^{\kappa}\to+\infty$ only if (6) is totally regular.* To show that (6) is not totally regular it would be sufficient to consider the sign of $a_{n,n-2}$ for large values of n.

We have by (7),

$$\begin{aligned}
a_{n,n-2}&=\sum_{r=n-2}^{n-1}\binom{r}{n-2}\left\{\left(1-\frac{r}{n}\right)^{\kappa}-\left(1-\frac{r+1}{n}\right)^{\kappa}\right\}\Delta^{r-n+2}(n-1)^{\kappa}\\
&=\left\{\left(1-\frac{n-2}{n}\right)^{\kappa}-\left(1-\frac{n-1}{n}\right)^{\kappa}\right\}(n-1)^{\kappa}+(n-1)\left\{\left(1-\frac{n-1}{n}\right)^{\kappa}-\left(1-\frac{n}{n}\right)^{\kappa}\right\}\times\Delta(n-1)^{\kappa}\\
&=\frac{1}{n^{\kappa}}\left[(2^{\kappa}-1)(n-1)^{\kappa}+(n-1)\{(n-1)^{\kappa}-n^{\kappa}\}\right]\\
&=\frac{1}{n^{\kappa}}\left[(2^{\kappa}-1)(n^{\kappa}-\kappa n^{\kappa-1}+\cdots)-\kappa(n-1)n^{\kappa-1}+\cdots\right]\\
&=\frac{1}{n^{\kappa}}\left[(2^{\kappa}-\kappa-1)n^{\kappa}+\text{terms containing lower powers of } n\right].
\end{aligned}$$

Hence, for sufficiently large n,

$$a_{n,n-2}<0,$$

since, for $0<\kappa<1$,

$$2^{\kappa}<\kappa+1.$$

Therefore, by a well known theorem (Hurwitz, p. 232, Theor. 1), the transformation (6) is not totally regular, so that $H_n^{\kappa}\to+\infty$ does not imply $\alpha_n^{\kappa}\to+\infty$.

Now suppose, if possible, that $H_n^{\kappa}\to+\infty$ implies $R^{\kappa}(\omega)\to+\infty$. Then since $R^{\kappa}(\omega)\to+\infty$ implies $\alpha_n^{\kappa}\to+\infty$, it follows that $H_n^{\kappa}\to+\infty$ implies $\alpha_n^{\kappa}\to+\infty$, which contradicts our previous result. This proves the theorem.

4. For the proof of Theorem 2 we require three lemmas. To establish these lemmas some properties of integral transformations will be needed. We begin with some definitions.

Given a series $\sum a_n$, suppose that the power series $\sum_{n=0}^{\infty}a_nx^n$ converges for $0<x<1$.

---

* $\| b_{n\nu} \|$ $(0\leq\nu\leq n)$ being a given matrix, the sequence-to-sequence transformation

$$\text{(B)}\qquad \sigma_n=\sum_{\nu=0}^{n}b_{n\nu}s_\nu \qquad (n=0,1,2,\ldots)$$

is said to be *regular*, if $s_n\to l$ always implies $\sigma_n\to l$. In case $\sigma_n\to l$ $(-\infty\leq l\leq+\infty)$ whenever $s_n\to l$, the transformation (B) is said to be *totally regular* (Hurwitz, 1927, p. 232).

We write

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = (1-x)\sum_{n=0}^{\infty} s_n x^n,$$

and call $A(x)$ the Abel mean. In case $\lim\limits_{x\to 1-0} A(x) = l\,(-\infty \leq l \leq +\infty)$, the series $\sum a_n$ is said to be summable (A) to l, *i.e.* summable by the Abel method to l. When for a series $\sum a_n$ the power series $\sum a_n x^n$ converges for $0 < x < 1$, we say that the (A) method *applies* to the series $\sum a_n$.

Let S denote the class of all functions $s(t)$ which are bounded and measurable in every finite interval $(0, \tau)$. Let $K(x, t)$ be a function defined in $0 < t < \tau$ for every $x > 0$, such that the integral

$$\sigma(x) = \int_0^{\infty} K(x, t)s(t)dt = \lim_{X\to\infty} \int_0^{X} K(x, t)s(t)dt \tag{8}$$

exists for every $x > 0$. Then (8) is a linear integral transformation defined by the *kernel* $K(x, t)$. When $\lim\limits_{X\to\infty} \sigma(x)$ exists and is equal to l, the kernel $K(x, t)$ defines a method of summability which we call K method or simply K. K or the transformation (8) is said to be *regular* (Agnew, 1939, *p.* 700) over the class S, if, for every $s(x) \in S$, $\lim\limits_{x\to\infty} s(x) = l$ implies that $\sigma(x)$ exists for every $x > 0$ and $\lim\limits_{x\to\infty} \sigma(x) = l$. We shall say that K is *totally regular,** in case it is regular and $\lim\limits_{x\to\infty} \sigma(x) = +\infty$ (or $-\infty$) whenever $\lim\limits_{x\to\infty} s(x) = +\infty$ (or $-\infty$).

When

$$K(x, t) \geq 0 \quad \text{for} \quad x, t > 0,$$

(8) is said to be a *positive kernal* tranformation.

The Cesàro mean $C^{\kappa}(x)$ of order $\kappa > 0$ for the function $s(x)$ is defined by the kernel

$$\begin{aligned} K(x, t) &= \frac{\kappa}{x^{\kappa}}(x-t)^{\kappa-1} \qquad && (0 \leqq t < x) \\ &= 0 && t \geq x; \end{aligned}$$

so that

$$C^{\kappa}(x) = \frac{\kappa}{x^{\kappa}} \int_0^{x} (x-t)^{\kappa-1} s(t)dt. \tag{9}$$

The Abel mean $a(x)$ for $s(x)$ is given by the transformation

$$a(x) = \frac{1}{x}\int_0^{\infty} e^{-t/x} s(t)dt, \tag{10}$$

the integral being supposed to exist† for every $x > 0$. When the integral (10) exists for every $x > 0$, we say that the Abel integral method or [A] method applies to the function $s(x)$.

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* In what follows by a function $s(x)$ we shall always mean one in the class S, and by regularity or total regularity we shall mean it over the class S

† Here we assume, for our purpose, that (10) exists as an absolutely convergent L-integral and not merely in the sense of (8)

Lemma 1. A sufficient condition that the regular* transformation (8) be totally regular is that

$$K(x, t) \geq 0$$

almost everywhere for $t \geq T$ and $x \geq X_1$.**

The proof of this lemma is similar to that of the sufficiency part of a known theorem (Basu (3), Theor. 1) and is therefore left out.

Lemma 2. With the notations of (9) and (10), for $\kappa > 0$, $C^\kappa(x) \to +\infty$ implies $a(x) \to +\infty$, provided that the [A] method applies.†

Proof. We have

$$\begin{aligned} a(x) &= \frac{1}{x}\int_0^\infty e^{-t/x}s(t)dt \\ &= \frac{1}{\Gamma(\kappa)x^{\kappa+1}}\int_0^\infty e^{-t/x}s(t)dt\int_0^\infty u^{\kappa-1}e^{-u/x}du \\ &= \frac{1}{\Gamma(\kappa)x^{\kappa+1}}\int_0^\infty e^{-u/x}du\int_0^u (u-t)^{\kappa-1}s(t)dt, \end{aligned}$$

the inversion being justified by absolute convergence; or

$$a(x) = \int_0^\infty K(x, t)C^\kappa(t)dt \quad \text{by (9)}, \tag{11}$$

where

$$K(x, t) = \frac{1}{\Gamma(\kappa+1)}\cdot\frac{1}{x}\left(\frac{t}{x}\right)^\kappa e^{-t/x} \qquad (x > 0).$$

The transformation (11) as defined by the kernel $K(x, t)$ is regular (See Knopp, 1941, p. 261); and since the kernel is positive, (11) is totally regular by Lemma 1. Lemma 2 now follows.

Lemma 3.‡ For $\kappa > 0$, if a series is summable (R, n, κ) to $+\infty$, then it is also (A) to $+\infty$, provided that the (A) method applies.

---

* As we have not used the conditions of regularity of K, they have not been stated here. For these conditions see Agnew, 1939, p. 700.

** This lemma is somewhat more general than what we actually require. It is an analogue of a well known theorem due to Hurwitz (Hurwitz, p. 247, Theor. VIII).

† It may be noted here that it has already been shown by Knopp that whereas, for $0 < \kappa \leq 1$, $C^\kappa(x) \to l$ implies that $a(x)$ exists for $x > 0$ and $a(x) \to l$ (Knopp, 1944, p. 155, Theor (1), when $\kappa > 1$, there is a function $s(x)$ for which $C^\kappa(x) \to l$, but the integral (10) does not exist for any $x > 0$ (Knopp, 1944, p. 156, Theor. 2); so that the hypothesis regarding the existence of the integral (10) has been necessary even for the validity of the crrresponding theorem for $\kappa > 1$, in the case of finite limits (See Knopp, 1944, p. 159).

‡ It may be noted here that a similar theorem holds for the (C, κ) summability; it is known that the corresponding theorem for the (H, κ) summability is false for $\kappa \geq 2$ (See Bosanquet, 1946, p 13) and the question whether or not, for $1 < \kappa < 2$, the (H, κ) summability of a series to $+\infty$ implies its (A) summability to $+\infty$ still remains open.

Proof. Given a series $\sum a_n$, suppose that the (A) method applies to it, *i.e.*,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

converges for $0 < z < 1$, or that $\sum_{n=0}^{\infty} a_n e^{-n/x}$ converges for $x > 0$, and defines the Abel mean

$$A(x) = \sum_{n=0}^{\infty} a_n e^{-n/x}.$$

Since $z = e^{-1/x}$, as $z \to 1-0$, $x \to \infty$.

Let

$$s(x) = \sum_{r=0}^{[x]} a_r ; \tag{12}$$

then

$$A(x) = \frac{1}{x}\int_0^{\infty} e^{-t/x} s(t)dt,$$

the integral converging absolutely for $x > 0$ (See e.g., Doetsch, 1943, p. 29, where the corresponding result is given for a Dirichlet series). Also, in this case, by (3) and (9),

$$C^{\kappa}(x) = R^{\kappa}(x).$$

Thus when $s(x)$ is restricted to a step function defined by (12), the Cesàro and Abel integral means for $s(x)$ reduce respectively to the Riesz and Abel matrix means for $\{s_n\}$. Lemma 3, therefore, follows immediately from Lemma 2.

Proof of Theorem 2. First suppose $1 < \kappa < 2$. Suppose, if possible, that $H_n^{\kappa} \to +\infty$ implies $R^{\kappa}(\omega) \to +\infty$. Then since, for $\kappa > 1$, $C_n^{\kappa} \to +\infty$ implies $H_n^{\kappa} \to +\infty$ (Basu (1), Theor. 3), it follows that, for $1 < \kappa < 2$, $C_n^{\kappa} \to +\infty$ implies $R^{\kappa}(\omega) \to +\infty$ which contradicts a known result (Basu (2), Theor. 2). Hence, for $1 < \kappa < 2$, $H_n^{\kappa} \to +\infty$ does not imply $R^{\kappa}(\omega) \to +\infty$.

Next suppose $\kappa \geq 2$. Suppose, if possible, that $H_n^{\kappa} \to +\infty$ implies $R^{\kappa}(\omega) \to +\infty$. Now, by Lemma 3, $R^{\kappa}(\omega) \to +\infty$ implies $A(x) \to +\infty$ provided that the (A) method is applicable. Consequently, for $\kappa \geq 2$, $H_n^{\kappa} \to +\infty$ implies $A(x) \to +\infty$ whenever the (A) method applies; but as we know this is false. The proof of Theorem 2 is now complete.

5. *Proof of Theorem 3.* In order to prove the last theorem it would be sufficient to show that, for $\kappa > 1$, there is a series for which

$$R^{\kappa}(\omega) \to +\infty,$$

while

$$\varliminf_{n\to\infty} H_n^{\kappa} \leq 0.$$

Here our method of proof is the same as that for the corresponding theorem for the (C, κ) method (See Basu (2), Theor. 4), but the details are now more complicated than before.

Consider the series $\sum_{n=0}^{\infty} a_n$ for which

$$a_0 = 1, \quad a_1 = -\alpha, \quad a_2 = \beta \quad \text{and} \quad a_n = 0 \text{ otherwise},$$

where α, β are positive numbers to be suitably chosen.

Since $\kappa > 1$, we can take α so that

$$\kappa+1 < \alpha < 2^{\kappa}. \tag{13}$$

We require one more lemma.

Lemma 4. α having been fixed subject to the condition (13), β can be chosen such that for the series $\sum a_n$,

$$\sigma^{\kappa}(\omega) > 0 \quad \text{for} \quad 0 < \omega < 3,$$

and

$$\sigma^{\kappa}(\omega) > 1 \quad \text{for} \quad \omega \geq 3.$$

For the proof of this lemma see Basu (2).

Next we define a series $\sum_{n=0}^{\infty} d_n$ by taking

$$d_{3n} = \lambda_n a_0, \quad d_{3n+1} = \lambda_n a_1 \quad \text{and} \quad d_{3n+2} = \lambda_n a_2 \quad (n = 0, 1, 2 \ldots), \tag{14}$$

where the λ's are positive numbers which we shall choose conveniently. Let $D^{\kappa}(\omega)$ be the Riesz sum of order κ for the series $\sum d_n$ and $dR^{\kappa}(\omega)$ end dH_n^{κ} its Riesz and Hölder means.

Then on putting $\omega/3 = n+\theta \quad (0 \leq \theta < 1)$,

$$D^{\kappa}(\omega) = \sum_{r=0}^{[\omega]} (\omega - r)^{\kappa} d_r = \sum_{\nu=0}^{n} \lambda_n \sigma^{\kappa}(\omega - 3\nu) > \lambda_{n-1} \quad \text{in virtue of Lemma 4.}$$

Hence

$$dR^{\kappa}(\omega) = \frac{D^{\kappa}(\omega)}{\omega^{\kappa}} > \frac{\lambda_{n-1}}{(3n+3\theta)^{\kappa}},$$

or

$$dR^{\kappa}(\omega) > \frac{\lambda_{n-1}}{3^{\kappa}(n+1)^{\kappa}}. \tag{15}$$

Again, by (1),

$$dH_{3n+1}^{\kappa} = \sum_{r=0}^{3n+1} \binom{3n+1}{r} \Delta^{3n+1-r} \mu_r . s_r', \tag{16}$$

where

$$s_r' = \sum_{m=0}^{r} d_m.$$

Using (14), we have

$$s_r' = \sum_{\nu=0}^{[r/3]} \lambda_\nu s_{r-3\nu}.$$

Hence from (16),

$$dH_{3n+1}^{\kappa} = \sum_{r=0}^{3n+1} \binom{3n+1}{r} \Delta^{3n+1-r} \mu_r \sum_{\nu=0}^{[r/3]} \lambda_\nu s_{r-3\nu}$$

$$= \sum_{\nu=0}^{n} \lambda^{\nu} \sum_{r=3\nu}^{3n+1} \binom{3n+1}{r} \Delta^{3n+1-r} \mu_r . s_{r-3\nu},$$

or

$$dH_{3n+1}^{\kappa} = \sum_{\nu=0}^{n} \lambda_\nu P_{n\nu}, \tag{17}$$

where

$$P_{n\nu} = \sum_{r=3\nu}^{3n+1} \binom{3n+1}{r} \Delta^{3n+1-r} \mu_r . s_{r-3\nu}.$$

Thus

$$P_{nn} = \sum_{r=3n}^{3n+1} \binom{3n+1}{r} \Delta^{3n+1-r} \mu_r . s_{r-3n} .$$

or, since

$$s_0 = 1, \quad s_1 = 1-\alpha,$$

$$\begin{aligned} P_{nn} &= (3n+1)(\mu_{3n}-\mu_{3n+1})+(1-\alpha)\mu_{3n+1} \\ &= (3n+1)\mu_{3n}-(3n+\alpha)\mu_{3n+1} \\ &= \frac{3n+1}{(3n+1)^\kappa} - \frac{3n+\alpha}{(3n+2)^\kappa} = \frac{\varphi(n)}{(3n+1)^\kappa(3n+2)^\kappa}, \text{ say.} \end{aligned}$$

Then

$$\begin{aligned} \varphi(n) &= (3n+1)(3n+2)^\kappa-(3n+\alpha)(3n+1)^\kappa \\ &= (3n+1)\{(3n)^\kappa+2\kappa(3n)^{\kappa-1}+ \ . \ . \ . \ \}-(3n+\alpha)\{(3n)^\kappa+\kappa(3n)^{\kappa-1}+ \ . \ . \ . \ \} \\ &= (\kappa+1-\alpha)(3n)^\kappa + \text{terms containing lower powers of } n. \end{aligned}$$

Hence, for sufficiently large n,

$$\varphi(n) < 0,$$

by (13). Accordingly, for sufficiently large n,

$$P_{nn} < 0. \tag{18}$$

Let us now choose the λ's such that

$$\lambda_{n-1} \geq (n+1)^{\kappa+1} \qquad (n = 1, 2, 3, \ . \ . \ . \ ), \tag{19}$$

and $\lambda_0, \lambda_1, \lambda_2, \ . \ . \ . \ , \lambda_{n-1}$ having been fixed subject to (19), we can always take λ_n sufficiently large such that from (17), in virtue of (18),

$$dH_{3n+1}^\kappa < 0,$$

for $n \geq N$, where N is large but fixed. Therefore, it follows that

$$\varliminf_{n\to} dH_n^\kappa \leq 0.$$

On the other hand, by (15) and (19),

$$dR^\kappa(\omega) > \frac{n+1}{3^\kappa}.$$

Hence, as $\omega = 3n+3\theta \to \infty$, $dR^\kappa(\omega) \to +\infty$.

This proves Theorem 3.

6. We append below a table embodying the results obtained in the present paper and in the two previous papers mentioned before, on the total relative strength of the Riesz, Cesàro and Hölder methods. From the table we can see at a glance how the three most commonly used methods of summability which are equivalent for finite limits (*i.e.*, if a series is summable to l by any of the three methods, then it is also summable to l by the other two) differ so widely from one another in the evaluation of divergent series, when the limits are infinite (*i.e.*, when a series is summable to $+\infty$ by any of these methods).

For simplicity we shall use some special symbols for this table. We denote the continuous matrix of the (R, n, κ) method for $\kappa \geq 0$ by R^κ, the matrices of the (C, κ) and

(H, κ) methods being denoted by C^κ and H^κ respectively for $\kappa > -1$. We shall use the symbol

$$H^\kappa \rightarrow C^\kappa$$

to mean that for a series, $H_n^\kappa \rightarrow +\infty$ implies $C_n^\kappa \rightarrow +\infty$ and when this is not the case, we write

$$H^\kappa \text{ not} \rightarrow C^\kappa.$$

TABLE

I. For $0 < \kappa < 1$*	II. For $\kappa > 1$
(1) C^κ to H^κ or R^κ	(4) C^κ to H^κ or R^κ
(i) C^κ not $\rightarrow H^\kappa$	(vii) $C^\kappa \rightarrow H^\kappa$
(ii) C^κ not $\rightarrow R^\kappa$	(viii) $\begin{cases} C^\kappa \rightarrow R^\kappa \ (\kappa = 2, 3, 4, \ldots) \\ C^\kappa \text{ not} \rightarrow R^\kappa \ \text{(otherwise)} \end{cases}$
(2) H^κ to C^κ or R^κ	(5) H^κ to C^κ or R^κ
(iii) $H^\kappa \rightarrow C^\kappa$	(ix) H^κ not $\rightarrow C^\kappa$
(iv) H^κ not $\rightarrow R^\kappa$	(x) H^κ not $\rightarrow R^\kappa$
(3) R^κ to C^κ or H^κ	(6) R^κ to C^κ or H^κ
(v) $R^\kappa \rightarrow C^\kappa$	(xi) R^κ not $\rightarrow C^\kappa$
(vi) $R^\kappa \rightarrow H^\kappa$ (?)	(xii) R^κ not $\rightarrow H^\kappa$
III. (7) $0 < \alpha < \beta < 1$ C^α not $\rightarrow H^\beta$	(8) $1 < \alpha < \beta$ H^α not $\rightarrow C^\beta$

(9) $-1 < \alpha \leq \beta < 0$

(i) $C^\alpha \rightarrow H^\alpha$

(ii) H^α not $\rightarrow C^\beta$.

Note. (*a*) (iv) includes (ii) in virtue of (iii).

(*b*) (viii), second part includes (x) for non-integral κ, in virtue of (vii).

(*c*) (xii) includes (xi) in virtue of (vii).

(*d*) (x) includes (ix) for $\kappa = 2, 3, 4, \ldots$, in virtue of (viii), first part.

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* We have omitted the trivial cases $\kappa = 0, 1$, when all the three matrices are *totally equivalent i.e.* if a series is summable to $l (-\infty \leq l \leq +\infty)$ by any one of the three methods, then it is also summable to l by the other two.

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ZUR ANALYSE DER LORENTZGRUPPE IN ALGEBRAISCHER DARSTELLUNG

Von

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Inhaltsübersicht: Fürs erste wird hier eine neue Ableitung des Einstein'schen Satzes über die Zerlegung der Lorentzgruppe mitgeteilt. Die Ableitung geht in algebraischer Darstellung vor und wird darum übersichtlicher. Ausserdem wird noch eine algebraische Darstellung der Lorentzgruppe dargebracht, welche sich als bedeutend einfach zeigt, weil sie auf dem alleinigen Gebrauch von Pauli-Matrizen beruht.

1. Einleitung.

Im Jahre 1932 liess Einstein und Mayer (1932) eine Arbeit veröffentlichen, welche für das Verständnis der Lorentztransformationen von grundlegender Bedeutung ist. Das für uns äusserst wichtige Ergebnis dieser Arbeit ist folgendes: Die Gruppe der reellen Lorentztransformationen lässt sich zerlegen in zwei gegeneinander vertauschbare Gruppen, die zueinander konjugiert komplex sind. Einige Jahre später vermochte Prof. S. N. Bose* (damals zu Dacca) den fertigen Satz von Einstein in algebraische Gestalt einzukleiden mittels *sechs* vierreihigen *reellen* Matrizen von besonderem Bau, deren drei sich gegen die übrigen vertauschen.

Noch später habe ich eine neue Ableitung des Einstein'schen Satzes gefunden, die sich etwa in drei Schritten vollzieht. Erstens zeige ich, dass eine infinitesimale Lorentztransformation in algebraische Gestalt gebracht werden kann mittels *drei* vierreihigen *komplexen* Matrizen und ihrer konjugierten. Zweitens spalte ich den algebraischen Ausdruck in zwei zueinander konjugiert komplexe Faktoren, welche sich gegeneinander vertauschen. Letztens wird gezeigt, dass sich eine beliebige endliche Lorentztransformation dadurch aufbauen lässt, dass man unendlich viele aufeinanderfolgende infinitesimale Lorentztransformationen zusammensetzt.

In der hier vorgelegten Mitteilung ist es meine Absicht zunächst diese Ableitung des Satzes von Einstein darzulegen und darnach noch eine algebraische Darstellung der reellen Lorentzgruppe darzubringen, bei der wir uns der wohlbekannten einfacheren zweireihigen Pauli-Matrizen bedienen und damit auch auskommen. Der Einfachheit halber kommt dieser letzteren eine ganz besondere Bedeutung zu.

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[*] Prof. Bose teilte mir derzeit den Befund brieflich mit. Sofern ich es weiss, ist der Befund seither nicht im Druck erschienen.

2. DIE MATRIX DER INFINITESIMALEN LORENTZTRANSFORMATION IN ALGEBRAISCHER KLEIDUNG UND IHRE SPALTUNG IN ZWEI ZUEINANDER KONJUGIERT KOMPLEXE FAKTOREN.

Es seien A^m, A_m die kontravarianten bzw-kovarianten Komponenten desselben Vierervektors. Zur infinitesimalen Transformation setzen wir

$$A'^m - A^m = dA^m = \varepsilon\Gamma^{m}_{\cdot s} A^s$$

bzw.

$$A'_m - A^m = dA_m = \varepsilon\Gamma^{\cdot s}_{m} A_s,$$

wo ε eine kleine Grösse und $\Gamma^{m}_{\cdot s}$, $\Gamma^{\cdot s}_{m}$ die Koeffizienten der Transformation seien. Da es sich bei der Transformation in der Tat um eine Drehung handelt, gilt

$$\begin{aligned} 0 &= d(A^m A_m), \\ &= \varepsilon\Gamma^{m}_{\cdot s} A^s A_m + A^m \varepsilon\Gamma^{\cdot s}_{m} A_s, \\ &= \varepsilon A^s A_m (\Gamma^{m}_{\cdot s} + \Gamma^{\cdot m}_{s}). \end{aligned}$$

Daher folgt zunächst

$$0 = \Gamma^{m}_{\cdot s} + \Gamma^{\cdot m}_{s},$$

dann aber mit Hilfe des Masstensors:

$$g_{44} = 1, \quad g_{11} = -1 = g_{22} = g_{33}, \quad g_{ms} = 0 \ (m \neq s),$$

erhält man

$$0 = \Gamma_{ms} + \Gamma_{sm}.$$

Daraus schliesst man, dass Γ_{ms} antisymmetrisch inbezug auf m, s ist. Die Matrix der infinitesimalen Lorentztransformation setzen wir sodann wie

$$d^{m}_{\cdot s} = \delta^{m}_{s} + \varepsilon\Gamma^{m}_{\cdot s} = \delta^{m}_{s} + \varepsilon g^{mm}\Gamma_{ms}$$

order ausgebreitet hingeschrieben wie

$$((d^{m}_{\cdot s})) = \begin{vmatrix} d^{4}_{\cdot 4}, & d^{4}_{\cdot 1}, & d^{4}_{\cdot 2}, & d^{4}_{\cdot 3} \\ d^{1}_{\cdot 4}, & d^{1}_{\cdot 1}, & d^{1}_{\cdot 2}, & d^{1}_{\cdot 3} \\ d^{2}_{\cdot 4}, & d^{2}_{\cdot 1}, & d^{2}_{\cdot 2}, & d^{2}_{\cdot 3} \\ d^{3}_{\cdot 4}, & d^{3}_{\cdot 1}, & d^{3}_{\cdot 2}, & d^{3}_{\cdot 3} \end{vmatrix} = \begin{vmatrix} 1, & \epsilon\Gamma_{41}, & \epsilon\Gamma_{42}, & \epsilon\Gamma_{43} \\ \epsilon\Gamma_{41}, & 1, & -\epsilon\Gamma_{12}, & \epsilon\Gamma_{31} \\ \epsilon\Gamma_{42}, & \epsilon\Gamma_{12}, & 1, & -\epsilon\Gamma_{23} \\ \epsilon\Gamma_{43}, & -\epsilon\Gamma_{31}, & \epsilon\Gamma_{23}, & 1 \end{vmatrix}$$

Behufs bequemeren Handhabung setzen wir

$$\begin{aligned} \alpha &= \tfrac{1}{2}(\Gamma_{14} + i\Gamma_{23}), \\ \beta &= \tfrac{1}{2}(\Gamma_{24} + i\Gamma_{31}), \\ \gamma &= \tfrac{1}{2}(\Gamma_{34} + i\Gamma_{12}), \end{aligned}$$

und haben demnach

$$((d^{m}_{\cdot s}))=\begin{vmatrix} 1\ , & -\epsilon(\alpha+\bar{\alpha}), & -\epsilon(\beta+\bar{\beta}), & -\epsilon(\gamma+\bar{\gamma}) \\ -\epsilon(\alpha+\bar{\alpha}), & 1\ , & i\epsilon(\gamma-\bar{\gamma})\ , & -i\epsilon(\beta-\bar{\beta}) \\ -\epsilon(\beta+\bar{\beta}), & -i\epsilon(\gamma-\bar{\gamma}), & 1\ , & i\epsilon(\alpha-\bar{\alpha}) \\ -\epsilon(\gamma+\bar{\gamma}), & i\epsilon(\beta-\bar{\beta})\ , & -i\epsilon(\alpha-\bar{\alpha}), & 1 \end{vmatrix}$$

$$= 1-\epsilon\alpha\tau_z-\epsilon\beta\tau_y-\epsilon\gamma\tau_z-\epsilon\bar{\alpha}\bar{\tau}_x-\epsilon\bar{\beta}\bar{\tau}_y-\epsilon\bar{\gamma}\bar{\tau}_z, \tag{1}$$

wobei wir die speziellen Matrizen, nämlich,

$$\tau_x=\begin{vmatrix} 0, & 1, & 0, & 0 \\ 1, & 0, & 0, & 0 \\ 0, & 0, & 0, & -i \\ 0, & 0, & i, & 0 \end{vmatrix}, \quad \tau_y=\begin{vmatrix} 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & i \\ 1, & 0, & 0, & 0 \\ 0, & -i, & 0, & 0 \end{vmatrix}, \quad \tau_z=\begin{vmatrix} 0, & 0, & 0, & 1 \\ 0, & 0, & -i, & 0 \\ 0, & i, & 0, & 0 \\ 1, & 0, & 0, & 0 \end{vmatrix}$$

und ihre konjugierten einführen. Damit wird die Matrix der infinitesimalen Lorentztransformation in algebraischer Kleidung erhalten mit Hilfe der drei speziellen Matrizen τ_x, τ_y, τ_z und ihrer konjugierten $\bar{\tau}_x$, $\bar{\tau}_y$, $\bar{\tau}_z$.

Die Spaltung der Matrix in zwei konjugiert komplexe Faktoren geschieht nun sofort wie

$$((d^{m}_{\cdot s})) = \left[e^{i\theta}\{1-(\alpha\tau_x+\beta\tau_y+\gamma\tau_z)\}\right]\cdot\left[e^{-i\theta}\{1-(\bar{\alpha}\bar{\tau}_x+\bar{\beta}\bar{\tau}_y+\bar{\gamma}\bar{\tau}_z)\}\right] \tag{2}$$

Der Grad der Näherung in (2) ist, wie man es merkt, der gleiche wie der der Matrix (1). Mit θ ist dabei eine reelle Zahl zu verstehen

3. WICHTIGE EIGENSCHAFTEN DER MATRIZEN (τ_x, τ_y, τ_z) UND $(\bar{\tau}_x, \bar{\tau}_y, \bar{\tau})$.

Durch einfache Multiplikation bestätigt man leicht die nachstehenden Eigenschaften:

(*i*) Jede Matrix der Art (τ_x, τ_y, τ_z) kommutiert mit jeder Matrix der Art $(\bar{\tau}_x, \bar{\tau}_y, \bar{\tau}_z)$ so, dass gelten

$$(\tau_x, \tau_y, \tau_z).(\bar{\tau}_x, \bar{\tau}_y, \tau_z) = (\bar{\tau}_x, \bar{\tau}_y, \bar{\tau}_z).(\tau_x, \tau_y, \tau_z) \tag{3}$$

um die Sache in kurzer Schreibweise auszudrücken.

(*ii*) Es gelten

$$\left.\begin{aligned} \tau_x^2 = 1 = \tau_y^2 &= \tau_z^2, \\ \tau_y\tau_z = i\tau_x &= -\tau_z\tau_y, \\ \tau_z\tau_x = i\tau_y &= -\tau_x\tau_z, \\ \tau_x\tau_y = i\tau_z &= -\tau_y\tau_x. \end{aligned}\right\} \tag{4}$$

für die Matrizen $(\bar\tau_x, \bar\tau_y, \bar\tau_z)$. Die entsprechenden konjugierten Gleichungen, nämlich,

$$\left.\begin{aligned} \bar\tau_x^2 = \quad 1 &= \bar\tau_y^2 = \bar\tau_z^2, \\ \bar\tau_y\bar\tau_z &= -\, i\bar\tau_x = -\, \bar\tau_z\bar\tau_y, \\ &\text{usw.} \end{aligned}\right\} \tag{5}$$

gelten für die Matrizen $(\bar\tau, \bar\tau_y, \bar\tau_z)$.

4. ZUSAMMENSETZUNG ZWEIER EVT. ENDLICH VIELER (n) INFINITESIMALER LORENTZTRANSFORMATIONEN.

Setzt man zwei infinitesimale Lorentztransformationen zusammen, so hat man für die Produktmatrix mit Hilfe von (2)

$$\begin{aligned} ((d^m_{\cdot s}))_2 \cdot ((d^m_{\cdot s}))_1 = {} & [e^{i\theta_2}\{1-\epsilon(\alpha_2\tau_x+\beta_2\tau_y+\gamma_2\tau_z)\}]\cdot[e^{-i\theta_2}\{1-(\bar\alpha_2\bar\tau_x \\ & +\bar\beta_2\bar\tau_y+\bar\gamma_2\bar\tau_z)\}]\cdot[e^{i\theta_1}\{1-\epsilon(\alpha_1\tau_x+\beta_1\tau_y+\gamma_1\tau_z)\}]\cdot[e^{-i\theta_1}\{1-\epsilon(\bar\alpha_1\bar\tau_x+\bar\beta_1\bar\tau_y+\bar\gamma_1\bar\tau_z)\}] \\ = {} & [e^{i(\theta_2+\theta_1)}\{1-\epsilon(\alpha_2\tau_x+\beta_2\tau_y+\gamma_2\tau_z)\}\{1-\epsilon(\alpha_1\tau_x+\beta_1\tau_y+\gamma_1\tau_z)\}]\cdot \\ & [e^{-i(\theta_2+\theta_1)}\{1-\epsilon(\bar\alpha_2\bar\tau_x+\bar\beta_2\bar\tau_y+\bar\gamma_2\bar\tau_z)\}\{1-(\bar\alpha_1\bar\tau_x+\bar\beta_1\bar\tau_y+\bar\gamma_1\bar\tau_z)\}] \end{aligned}$$

mit Rücksicht auf die Vertauschbarkeit [Gl. (3)] von (τ_x, τ_y, τ_z) und $(\bar\tau_x, \bar\tau_y, \tau_z)$ gegeneinander. Setzt man nun zur Abkürzung mit Rücksicht auf die Multiplikationsgesetze [Gln. (4) u (5)] von (τ_x, τ_y, τ_z) bzw. $(\bar\tau_x, \bar\tau_y, \bar\tau_z)$

$$e^{i(\theta_2+\theta_1)}\{1-\epsilon(\alpha_2\tau_x+\beta_2\tau_y+\gamma_2\tau_z)\}\{1-\epsilon(\alpha_1\tau_x+\beta_1\tau_y+\gamma_1\tau_z)\} = d_2-a_2\tau_x-b_2\tau_y-c_2\tau_z$$

und

$$e^{-i(\theta_2+\theta_1)}\{1-\epsilon(\bar\alpha_2\bar\tau_x+\bar\beta_2\bar\tau_y+\bar\gamma_2\bar\tau_z)\}\{1-\epsilon(\bar\alpha_1\bar\tau_x+\bar\beta_1\bar\tau_y+\bar\gamma_1\bar\tau_z)\} = \bar d_2-\bar a_2\bar\tau_x-\bar b_2\bar\tau_y-\bar c_2\bar\tau_z,$$

so findet man ohne Mühe

$$d_2^2-a_2^2-b_2^2-c_2^2 = e^{2i(\theta_2+\theta_1)}\{1-\epsilon^2(\alpha_2^2+\beta_2^2+\gamma_2^2)\}\cdot\{1-\epsilon^2(\alpha_1^2+\beta_1^2+\gamma_1^2)\}$$

und

$$\bar d_2^2-\bar a_2^2-\bar b_2^2-\bar c_2^2 = e^{-2i(\theta_2+\theta_1)}\{1-\epsilon^2(\bar\alpha_2^2+\bar\beta_2^2+\bar\gamma_2)\}\cdot\{1-\epsilon^2(\bar\alpha_1^2+\bar\beta_1^2+\bar\gamma_1^2)\}$$

Setzt man nun dem erhaltenen Produkte noch eine dritte infinitesimale Lorentztransformation hinzu, so hat man für das Produkt

$$\begin{aligned} ((d^m_{\cdot s}))_3 \cdot ((d^m_{\cdot s}))_2 \cdot ((d^m_{\cdot s}))_1 = {} & [e^{i\theta_3}\{1-\epsilon(\alpha_3\tau_x+\beta_3\tau_y+\gamma_3\tau_z)\}]\cdot \\ & \cdot[e^{-i\theta_3}\{1-\epsilon(\bar\alpha_3\bar\tau_x+\bar\beta_3\bar\tau_y+\bar\gamma_3\bar\tau_z)\}]\cdot(d_2-a_2\tau_x-b_3\tau_y-c_2\tau_z)(\bar d_2-\bar a_2\bar\tau_x-\bar b_2\bar\tau_y-\bar c_2\bar\tau_z) \\ = {} & [e^{i\theta_3}\{1-\epsilon(\alpha_3\tau_x+\beta_3\tau_y+\gamma_3\tau_z)\}](d_2-a_2\tau_x-b_2\tau_y-c_2\tau_z) \\ & \cdot[e^{-i\theta_3}\{1-\epsilon(\bar\alpha_3\bar\tau_x+\bar\beta_3\bar\tau_y+\bar\gamma_3\bar\tau_z)\}](\bar d_2-\bar a_2\bar\tau_x-\bar b_2\bar\tau_y-\bar c_2\bar\tau_z) \\ = {} & (d_3-a_3\tau_x-b_3\tau_y-c_3\tau_z)(\bar d_3-\bar a_3\bar\tau_x-\bar b_3\bar\tau_y-\bar c_3\bar\tau_z) \end{aligned}$$

zur Abkürzung. Man findet dann ebenso wie oben

$$d_3^2-a_3^2-b_3^2-c_3^2 = e^{2i\theta_3}\{1-\epsilon^2(\alpha_3^2+\beta_3^2+\gamma_3^2)\}(d_2^2-a_2^2-b_2^2-c_2^2)$$

$$= e^{2i(\theta_3+\theta_2+\theta_1)}\{1-\epsilon^2(\alpha_3^2+\beta_3^2+\gamma_3^2)\}\cdot\{1-\epsilon^2(\alpha_2^2+\beta_2^2+\gamma_2^2)\}\{1-\epsilon^2(\alpha_1^2+\beta_1^2+\gamma_1^2)\}$$

und

$$\bar{d}_3^2-\bar{a}_3^2-\bar{b}_3^2-\bar{c}_3^2 = e^{-2i(\theta_3+\theta_2+\theta_1)}\{1-\epsilon^2(\bar{\alpha}_3^2+\bar{\beta}_3^2+\bar{\gamma}_3^2)$$

$$\cdot\{1-\epsilon^2(\bar{\alpha}_2^2+\bar{\beta}_2^2+\bar{\gamma}_2^2)\}\{1-\epsilon^2(\bar{\alpha}_1^2+\bar{\beta}_1^2+\bar{\gamma}_1^2)\}$$

Verfährt man in dieser Weise weiter, so findet man für den Fall von n aufeinanderfolgenden infinitesimalen Lorentztransformationen das Produkt wie

$$\begin{aligned}((d_{\cdot s}^{m}))_n\cdots((d_{\cdot s}^{m}))_2((d_{\cdot s}^{m}))_1 = {} & [e^{i(\theta+\cdots+\theta_n)}\{1-\epsilon(\alpha_n\tau_x+\beta_n\tau_y+\gamma_n\tau_z)\} \\ & \cdots\{1-\epsilon(\alpha_2\tau_x+\beta_2\tau_y+\gamma_2\tau_z)\}\{1-\epsilon(\alpha_1\tau_x+\beta_1\tau_y+\gamma_1\tau_z)\}] \\ & \cdot[e^{-i(\theta_1+\cdots+\theta_n)}\{1-\epsilon(\bar{\alpha}_n\bar{\tau}_x+\bar{\beta}_n\bar{\tau}_y+\bar{\gamma}_n\bar{\tau}_z)\}\cdots \\ & \cdots\{1-\epsilon(\bar{\alpha}_2\bar{\tau}_x+\bar{\beta}_2\bar{\tau}_y+\bar{\gamma}_2\bar{\tau}_z)\}\{1-\epsilon(\bar{\alpha}_1\bar{\tau}_x+\bar{\beta}_1\bar{\gamma}_y+\bar{\gamma}_1\bar{\tau}_z)\}] \\ = {} & (d_n-a_n\tau_x-b_n\tau_y-c_n\tau_z)(\bar{d}_n-\bar{a}_n\bar{\tau}_x-\bar{b}_n\bar{\tau}_y-\bar{c}_n\bar{\tau}_z),\end{aligned} \tag{6}$$

wobei gelten

$$\left.\begin{aligned}d_n^2-a_n^2-b_n^2-c_n^2 = {} & e^{2i(\theta_1+\cdots+\theta_n)}\{1-\epsilon^2(\alpha_n^2+\beta_n^2+\gamma_n^2)\} \\ & \cdots\{1-\epsilon^2(\alpha_2^2+\beta_2^2+\gamma_2^2)\}\{1-\epsilon^2(\alpha_1^2+\beta_1^2+\gamma_1^2)\} \\ \bar{d}_n^2-\bar{a}_n^2-\bar{b}_n^2-\bar{c}_n^2 = {} & e^{-2i(\theta_1+\cdots+\theta_n)}\{1-\epsilon^2(\bar{\alpha}_n^2+\bar{\beta}_n^2+\bar{\gamma}_n^2)\} \\ & \cdots\{1-\epsilon^2(\bar{\alpha}_2^2+\bar{\beta}_2^2+\bar{\gamma}_2^2)\}\{1-\epsilon^2(\bar{\alpha}_1^2+\bar{\beta}_1^2+\bar{\gamma}_1^2)\}\end{aligned}\right\} \tag{7}$$

5. AUFBAU EINER BELIEBIGEN ENDLICHEN LORENTZTRANSFORMATION DURCH ZUSAMMENSETZUNG UNENDLICH VIELER INFINITESIMALER LORENTZTRANSFORMATIONEN

Setzt man in den Formeln (6) u (7)

$$(\alpha_1, \beta_1, \gamma_1) = (\alpha_2, \beta_2, \gamma_2) = \cdots = (\alpha_n, \beta_n, \gamma_n = (\alpha, \beta, \gamma)$$

und zur Abkürzung noch

$$\alpha\tau_x+\beta\tau_y+\gamma\tau_z = T$$

so erhält man an deren Stelle

$$[e^{i(\theta_1+\cdots+\theta_n)}(1-\epsilon T)^n]\cdot[e^{-i(\theta_1+\cdots+\theta_n)}.(1-\epsilon\bar{T})^n]$$

$$= (d_n-a_n\tau_x-b_n\tau_y-c_n\tau_z)\cdot(\bar{d}_n-\bar{a}_n\bar{\tau}_x-\bar{b}_n\bar{\tau}_y-\bar{c}_n\bar{\tau})$$

und

$$\left.\begin{aligned}d_n^2-a_n^2-b_n^2-c_n^2 &= e^{2i(\theta_1+\cdots+\theta_n)}.\{1-\epsilon^2(\alpha_2^2+\beta^2+\gamma^2)\}^n \\ \bar{d}_n^2-\bar{a}_n^2-\bar{b}_n^2-\bar{c}_n^2 &= e^{-2i(\theta_1+\cdots+\theta_n)}.\{1-\epsilon^2(\bar{\alpha}^2+\bar{\beta}^2+\bar{\gamma}^2)\}^n\end{aligned}\right\}$$

Geht man nun zum Grenzfall:

$$\left.\begin{array}{l}\epsilon \rightarrow 0\\ n \rightarrow \infty\end{array}\right\} \text{mit } n\epsilon = 1$$

über, so erhält man mit der Bezeichnung:

$$\left.\begin{array}{l}(d_n,\ a_n,\ b_n,\ c_n)_{n\to\infty} = (d,\ a,\ b,\ c)\\ (\bar{d}_n,\ \bar{a}_n,\ \bar{b}_n,\ \bar{c}_n)_{n\to\infty} = (\bar{d},\ \bar{a},\ \bar{b},\ \bar{c})\end{array}\right\}$$

und mit dem Ansatz:

$$(\theta_1 + \cdots + \theta_n) = 2m\pi + \delta,$$

wo m eine ganze Zahl ist und $|\delta| \leq \pi$,

$$\left.\begin{array}{r}d - a\tau_x - b\tau_y - c\tau_z = e^{i\delta}.\ \text{Limes}\ (1-\epsilon T)^n\\ = \exp\,(-T + i\delta)\\ \bar{d} - \bar{a}\bar{\tau}_x - \bar{b}\bar{\tau}_y - \bar{c}\bar{\tau}_z = \exp\,(-\bar{T} - i\delta)\end{array}\right\} \tag{8}$$

und

$$\left.\begin{array}{l}d^2 - a^2 - b^2 - c^2 = \exp\,(2i\delta)\\ \bar{d}^2 - \bar{a}^2 - \bar{b}^2 - \bar{c}^2 = \exp\,(-2i\delta)\end{array}\right\} \tag{9}$$

Wie man es merkt, lassen die Gleichungen (8) mit Hilfe der Multiplikationsgesetze von (τ_x, τ_y, τ_z) bzw. $(\bar{\tau}_x, \bar{\tau}_y, \bar{\tau}_z)$ [Gln. (4) bzw. (5)] die Grössen (d, a, b, c) bzw. $(\bar{d}, \bar{a}, \bar{b}, \bar{c})$ in Funktion von (α, β, γ) bzw. $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ explizite ausdrücken. Die Ausdrücke findet man ja leicht wie

$$d = e^{i\delta} \cosh\,(\alpha^2 + \beta^2 + \gamma^2)^{\frac{1}{2}}$$

$$(a,\ b,\ c) = e^{i\delta} \sinh\,(\alpha^2 + \beta^2 + \gamma^2)^{\frac{1}{2}}.(\alpha,\ \beta,\ \gamma)/(\alpha^2 + \beta^2 + \gamma^2)^{\frac{1}{2}}$$

usw. Wie es also zu erwarten ist, sind die Grössen (d, a, b, c) bzw. $(\bar{d}, \bar{a}, \bar{b}, \bar{c})$ nur der Bedingung (9) unterworfen, sonst aber beliebig.

Dementsprechend fassen wir das Endergebnis unserer bisherigen Untersuchung folgendermassen zusammen: Die Matrix einer beliebigen endlichen Lorentztransformation lässt sich ausdrücken wie

$$\boxed{\begin{array}{c}(d - a\tau_x - b\tau_y - c\tau_z)(\bar{d} - \bar{a}\bar{\tau}_x - \bar{b}\bar{\tau}_y - \bar{c}\bar{\tau}_z)\\ \text{mit}\\ \tau_x = \begin{vmatrix} 0, & 1, & 0, & 0\\ 1, & 0, & 0, & 0\\ 0, & 0, & 0, & -i\\ 0, & 0, & i, & 0\end{vmatrix},\quad \tau_y = \begin{vmatrix} 0, & 0, & 1, & 0\\ 0, & 0, & 0, & i\\ 1, & 0, & 0, & 0\\ 0, & -i, & 0, & 0\end{vmatrix},\quad \tau_z = \begin{vmatrix} 0, & 0, & 0, & 1\\ 0, & 0, & -i, & 0\\ 0, & i, & 0, & 0\\ 1, & 0, & 0, & 0\end{vmatrix}\\ \text{und}\\ \left.\begin{array}{l}d^2 - a^2 - b^2 - c^2 = e^{2i\delta}\\ \bar{d}^2 - \bar{a} - \bar{b}^2 - \bar{c}^2 = e^{-2i\delta}\end{array}\right\}\end{array}} \tag{10}$$

Wir wollen nun anmerken, dass die der obenstehenden inverse Transformation anzugeben ist wie

$$(d + a\tau_x + b\tau_y + c\tau_z)(\bar{d} + \bar{a}\bar{\tau}_x + \bar{b}\bar{\tau}_y + \bar{c}\bar{\tau}_z),$$

weil das Produkt beider

$$\begin{aligned}
&= (d - a\tau_x - b\tau_y - c\tau_z)(\bar{d} - \bar{a}\bar{\tau}_x - \bar{b}\bar{\tau}_y - \bar{c}\bar{\tau}_z)\cdot(d + a\tau_x + b\tau_y + c\tau_z)(\bar{d} + \bar{a}\bar{\tau}_x + \bar{b}\bar{\tau}_y + \bar{c}\bar{\tau}_z)\\
&= (d - a\tau_x - b\tau_y - c\tau_z)(d + a\tau_x + b\tau_y + c\tau_z)\cdot(\bar{d} - \bar{a}\bar{\tau} - \bar{b}\bar{\tau}_y - \bar{c}\bar{\tau}_z)(\bar{d} + \bar{a}\bar{\tau}_x + \bar{b}\bar{\tau}_y + \bar{c}\bar{\tau}_z)\\
&= (d^2 - a^2 - b^2 - c^2)(\bar{d}^2 - \bar{a}^2 - \bar{b}^2 - \bar{c}^2)\\
&= 1
\end{aligned}$$

ist mit Hilfe von (10).

6. DIE LORENTZGRUPPE UND IHRE FAKTORGRUPPEN

Betrachtet man zwei endliche Lorentztransformationen, etwa

$$(d_1 + a_1\tau_x + b_1\tau_y + c_1\tau_z)(\bar{d}_1 + \bar{a}_1\bar{\tau}_x + \bar{b}_1\bar{\tau}_y + \bar{c}_1\bar{\tau}_z)$$

und

$$(d_2 + a_2\tau_x + b_2\tau_y + c_2\tau_z)(\bar{d}_2 + \bar{a}_2\bar{\tau}_x + \bar{b}_2\bar{\tau}_y + \bar{c}\bar{\tau}_z),$$

so ist das Produkt der beiden

$$\begin{aligned}
&= (d_2 + a_2\tau_x + b_2\tau_y + c_2\tau_z)(d_1 + a_1\tau_x + b_1\tau_y + c_1\tau_z)\\
&\qquad\cdot(\bar{d}_2 + \bar{a}_2\bar{\tau}_x + \bar{b}_2\bar{\tau}_y + \bar{c}_2\bar{\tau}_z)(\bar{d}_1 + \bar{a}_1\bar{\tau}_x + \bar{b}_1\bar{\tau}_y + \bar{c}_1\bar{\tau}_z)\\
&= (d' + a'\tau_x + b'\tau_y + c'\tau_z)(\bar{d}' + \bar{a}'\bar{\tau}_x + \bar{b}'\bar{\tau} + \bar{c}'\bar{\tau}_z)
\end{aligned}$$

gesetzt. Dann ist

$$\begin{aligned}
d'^2 - a'^2 - b'^2 - c'^2 &= (d_1d_2 + a_1a_2 + b_1b_2 + c_1c_2)^2 - \{(d_2a_1 + d_1a_2) + i(b_2c_1 - b_1c_2)\}^2 - \dots,\\
&= (d_1^2 - a_1^2 - b_1^2 - c_1^2)(d_2^2 - a_2^2 - b_2^2 - c_2^2)
\end{aligned}$$

und ebenso

$$\bar{d}'^2 - \bar{a}'^2 - \bar{b}'^2 - \bar{c}'^2 = (\bar{d}_1^2 - \bar{a}_1^2 - \bar{b}_1^2 - \bar{c}_1^2)(\bar{d}_2^2 - \bar{a}_2^2 - \bar{b}_2^2 - \bar{c}).$$

Daraus folgt, dass die Produkttransformation wieder eine Lorentz'sche ist, weil mit Hilfe von (10)

$$(d'^2 - a'^2 + b'^2 - c'^2)(\bar{d}'^2 - \bar{a}'^2 - \bar{b}'^2 - \bar{c}'^2) = 1$$

ist. Im allgemeinen ist also die Bedingung:

$$(d^2 - a^2 - b^2 - c^2)(\bar{d}^2 - \bar{a}^2 - \bar{b}^2 - \bar{c}^2) = 1 \tag{11}$$

entscheidend für die Lorentzgruppe.

Beschränkt man sich hingegen auf die Betrachtung der Faktortransformationen, etwa,

$$d_1 + a_1\tau_x + b_1\tau_y + c_1\tau_z,$$

und

$$d_2 + a_2\tau_x + b_2\tau_y + c_2\tau_z$$

so ist das Produkt beider, wie schon oben gezeigt,

$$= d' + a'\tau_x + b'\tau_y + c'\tau_z,$$

wobei gilt

$$d'^2 - a'^2 - b'^2 - c'^2 = (d_1^2 - a_1^2 - b_1^2 - c_1^2)(d_2^2 - a_2^2 - b_2^2 - c).$$

Mit Hilfe von (10) ist aber dann

$$d'^2 - a'^2 - b'^2 - c'^2 = e^{2i(\delta_1+\delta_2)} = e^{2i\delta'}.$$

Entscheidend für die Faktorgruppen ist im allgemeinen also die Bedingung

$$\left.\begin{aligned} d^2 - a^2 - b^2 - c^2 &= e^{2i\delta}, \\ \bar{d}^2 - \bar{a}^2 - \bar{b}^2 - \bar{c}^2 &= e^{-2i\delta}. \end{aligned}\right\} \tag{12}$$

7. DIE ALGEBRAISCHE DARSTELLUNG DER LORENTZGRUPPE MITTELS PAULI-MATRIZEN

Das Einsetzen der Ausdrücke von (τ_x, τ_y, τ_z) ergibt

$$d + a\tau_x + b\tau_y + c\tau_z = \begin{vmatrix} d, & a, & b, & c \\ a, & d, & -ic, & ib \\ b, & ic, & d, & -ia \\ c, & -ib, & ia, & d \end{vmatrix}$$

Eine beliebige Lorentztransformation schreibt man hin demnach wie

$$\begin{vmatrix} y^{(4)} \\ y^{(1)} \\ y^{(2)} \\ y^{(3)} \end{vmatrix} = \begin{vmatrix} d, & a, & b, & c \\ a, & d, & -ic, & ib \\ b, & ic, & d, & -ia \\ c, & -ib, & ia, & d \end{vmatrix} \begin{vmatrix} \bar{d}, & \bar{a}, & \bar{b}, & \bar{c} \\ \bar{a}, & \bar{d}, & i\bar{c}, & -i\bar{b} \\ \bar{b}, & -i\bar{c}, & \bar{d}, & i\bar{a} \\ \bar{c}, & i\bar{b}, & -i\bar{a}, & \bar{d} \end{vmatrix} \begin{vmatrix} x^{(4)} \\ x^{(1)} \\ x^{(2)} \\ x^{(3)} \end{vmatrix}$$

Wir denken uns nun die Lorentztransformation mit zwei Schritten ausgeführt, wie etwa

$$\begin{vmatrix} y^{(4)} \\ y^{(1)} \\ y^{(2)} \\ y^{(3)} \end{vmatrix} = \begin{vmatrix} d, & a, & b, & c \\ a, & d, & -ic, & ib \\ b, & ic, & d, & -ia \\ c, & -ib, & ia, & d \end{vmatrix} \begin{vmatrix} \varphi^{(4)} \\ \varphi^{(1)} \\ \varphi^{(2)} \\ \varphi^{(3)} \end{vmatrix} \tag{13}$$

für den ersten Schritt und

$$\begin{vmatrix} \varphi^{(4)} \\ \varphi^{(1)} \\ \varphi^{(2)} \\ \varphi^{(3)} \end{vmatrix} = \begin{vmatrix} \bar{d}, & \bar{a}, & \bar{b}, & \bar{c} \\ \bar{a}, & \bar{d}, & i\bar{c}, & -i\bar{b} \\ \bar{b}, & -i\bar{c}, & \bar{d}, & i\bar{a} \\ \bar{c}, & i\bar{b}, & -i\bar{a}, & \bar{d} \end{vmatrix} \begin{vmatrix} x^{(4)} \\ x^{(1)} \\ x^{(2)} \\ x^{(3)} \end{vmatrix} \tag{14}$$

für den zweiten. Aus Gleichung (13) folgt

$$\left.\begin{aligned} y^{(4)}+y^{(3)} &= (d+c)(\varphi^{(4)}+\varphi^{(3)}) + (a-ib)(\varphi^{(1)}+i\varphi^{(2)}) \\ y^{(4)}-y^{(3)} &= (d-c)(\varphi^{(4)}-\varphi^{(3)}) + (a+ib)(\varphi^{(1)}-i\varphi^{(2)}) \\ y^{(1)}-iy^{(2)} &= (a-ib)(\varphi^{(4)}-\varphi^{(3)}) + (d+c)(\varphi^{(1)}-i\varphi^{(2)}) \\ y^{(1)}+iy^{(2)} &= (a+ib)(\varphi^{(4)}+\varphi^{(3)}) + (d-c)(\varphi^{(1)}+i\varphi^{(2)}) \end{aligned}\right\}$$

d.h.

$$\begin{vmatrix} y^{(4)}+y^{(3)}, & y^{(1)}-iy^{(2)} \\ y^{(1)}+iy^{(2)}, & y^{(4)}-y^{(3)} \end{vmatrix} = \begin{vmatrix} d+c, & a-ib \\ a+ib, & d-c \end{vmatrix} \begin{vmatrix} \varphi^{(4)}+\varphi^{(3)}, & \varphi^{(1)}-i\varphi^{(2)} \\ \varphi^{(1)}+i\varphi^{(2)}, & \varphi^{(4)}-\varphi^{(3)} \end{vmatrix}$$

oder

$$(y^{(4)}+y^{(1)}\sigma_x+y^{(2)}\sigma_y+y^{(3)}\sigma_z) = (d+a\sigma_x+b\sigma_y+c\sigma_z)(\varphi^{(4)}+\varphi^{(1)}\sigma_x+\varphi^{(2)}\sigma_y+\varphi^{(3)}\sigma_z) \tag{15}$$

mit Hilfe der Pauli-Matrizen

$$\sigma_x = \begin{vmatrix} 0, & 1 \\ 1, & 0 \end{vmatrix}, \quad \sigma_y = \begin{vmatrix} 0, & -i \\ i, & 0 \end{vmatrix}, \quad \sigma_z = \begin{vmatrix} 1, & 0 \\ 0, & -1 \end{vmatrix}$$

Eine gleiche Behandlung der Gleichung (14) ergibt zunächst

$$\begin{vmatrix} \varphi^{(4)}+\varphi^{(3)}, & \varphi^{(1)}-i\varphi^{(2)} \\ \varphi^{(1)}+i\varphi^{(2)}, & \varphi^{(4)}-\varphi^{(3)} \end{vmatrix} = \begin{vmatrix} x^{(4)}+x^{(3)}, & x^{(1)}-ix^{(2)} \\ x^{(1)}+ix^{(2)}, & x^{(4)}-x^{(3)} \end{vmatrix} \begin{vmatrix} \bar{d}+\bar{c}, & \bar{a}-i\bar{b} \\ \bar{a}+i\bar{b}, & \bar{d}-\bar{c} \end{vmatrix}$$

und dann

$$(\varphi^{(4)}+\varphi^{(1)}\sigma_x+\varphi^{(2)}\sigma_y+\varphi^{(3)}\sigma_z) = (x^{(4)}+x^{(1)}\sigma_x+x^{(2)}\sigma_y+x^{(3)}\sigma_z)(\bar{d}+\bar{a}\sigma_x+\bar{b}\sigma_y+\bar{c}\sigma_z) \tag{16}$$

Die Verbindung der Glh. (15) u (16) liefert für die volle Lorentztransformation die Formel

$$\boxed{\begin{aligned} (y^{(4)}+y^{(1)}\sigma_x+y^{(2)}\sigma_y+y^{(3)}\sigma_z) &= (d+a\sigma_x+b\sigma_y+c\sigma_z) \\ &\cdot(x^{(4)}+x^{(1)}\sigma_x+x^{(2)}\sigma_y+x^{(3)}\sigma_z) \\ &\cdot(\bar{d}+\bar{a}\sigma_x+\bar{b}\sigma_y+\bar{c}\sigma_z) \end{aligned}} \tag{17}$$

Über die Gruppenstruktur entscheiden auch jetzt, wie das zu merken ist: die Bedingungen (11) bzw. (12) für die volle Gruppe bzw. die Faktorgruppen.

Tauscht man die Schritten um, etwa wie

$$\begin{vmatrix} y^{(4)} \\ y^{(1)} \\ y^{(2)} \\ y^{(3)} \end{vmatrix} = \begin{vmatrix} \bar{d}, & \bar{a}, & \bar{b}, & \bar{c} \\ \bar{a}, & \bar{d}, & i\bar{c}, & -i\bar{b} \\ \bar{b}, & -i\bar{c}, & \bar{d}, & i\bar{a} \\ \bar{c}, & i\bar{b}, & -i\bar{a}, & \bar{d} \end{vmatrix} \begin{vmatrix} \varphi^{(4)} \\ \varphi^{(1)} \\ \varphi^{(2)} \\ \varphi^{(3)} \end{vmatrix}$$

oder

$$(y^{(4)}+y^{(1)}\sigma_x+y^{(2)}\sigma_y+y^{(3)}\sigma_z) = (\varphi^{(4)}+\varphi^{(1)}\sigma_x+\varphi^{(2)}\sigma_y+\varphi^{(3)}\sigma_z)(\bar{d}+\bar{a}\sigma_x+\bar{b}\sigma_y+\bar{c}\sigma_z)$$

für den ersten Schritt und

$$\begin{vmatrix} \varphi^{(4)} \\ \varphi^{(1)} \\ \varphi^{(2)} \\ \varphi^{(3)} \end{vmatrix} = \begin{vmatrix} d, & a, & b, & c \\ a, & d, & -ic, & ib \\ b, & ic, & d, & -ia \\ c, & -ib, & ia, & d \end{vmatrix} \begin{vmatrix} x^{(4)} \\ x^{(1)} \\ x^{(2)} \\ x^{(3)} \end{vmatrix}$$

oder

$$(\varphi^{(4)} + \varphi^{(1)}\sigma_x + \varphi^{(2)}\sigma_y + \varphi^{(3)}\sigma_z) = (d + a\sigma_x + b\sigma_y + c\sigma_z)(x^{(1)} + x^{(1)}\sigma_x + x^{(2)}\sigma_y + x^{(3)}\sigma_z)$$

für den zweiten, so ergibt die Verbindung beider die gleiche Formel (17) für die volle Lorentztransformation. Es hängt dies offenbar damit zusammen, dass die Matrizen

$$d + a\tau_x + b\tau_y + c\tau_z,$$

$$\bar{d} + \bar{a}\bar{\tau}_x + \bar{b}\bar{\tau}_y + \bar{c}\bar{\tau}_z$$

gegeneinander vertauschbar sind.

8. NACHTRAG

Nachträglich sehe ich ein, dass die Ableitung der Formel (10) auch anders, dabei etwas kürzer, erfolgen dürfte, falls man folgendermassen vorginge. Man nehme, nämlich, die Formel (1)' für die volle infinitesimale Lorentztransformation in Betracht, setze sodann n Wiederholungen derselben zusammen und erhalte somit beim Grenzfall:

$$\left.\begin{array}{l} \epsilon \rightarrow 0 \\ n \rightarrow \infty \end{array}\right\} \text{ mit } n\epsilon = 1,$$

die Matrix der endlichen Lorentztransformation wie

$$\text{Limes } \{1 - \epsilon(T + \bar{T})\}^n = \exp(-T - \bar{T}),$$

wo T, $\bar{T}$ dieselbe Bedeutungen haben wie früher. Erst dann spalte man die Matrix auf wie

$$\exp(-T + i\delta) \cdot \quad \exp(-\bar{T} - i\delta)$$

$$= (d - a\tau_x - b\tau_y - c\tau_z) \cdot (\bar{d} - \bar{a}\bar{\tau}_x - \bar{b}\bar{\tau}_y - \bar{c}\bar{\tau}_z)$$

Mit Hilfe der expliziten Ausdrücke von (d, a, b, c) bzw. $(\bar{d}, \bar{a}, \bar{b}, \bar{c})$ zeige man dann

$$\left.\begin{array}{l} d^2 - a^2 - b^2 - c^2 = e^{2i\delta} \\ \bar{d}^2 - \bar{a}^2 - \bar{b}^2 - \bar{c}^2 = e^{-2i\delta} \end{array}\right\}$$

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ON THE FLEXURE PROBLEM FOR SOME BOUNDARIES

By

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1. The general problem of flexure for an isotropic elastic cylinder on the terminal cross-section of which is placed a load in any direction in the plane of the cross-section can be resolved into two simpler problems: (1) the problem of a pure flexure (when the torsion is nil) when the load is supposed to act through a definite point on the terminal cross-section called the centre of flexure (Stevenson 1938), parallel to its original direction and (2) the problem of a pure torsion by a couple whose moment is equal to the moment of the load about the centre of flexure. The problem of pure torsion has been treated by Muschelisvili (1929) and the problem of flexure has been treated by applying the function-theoretic method by Ghosh (1947). The method of solution in the above two cases, by the authors, requires primarily the determination of a suitable mapping function which makes a conformal representation of the cross-section of the cylinder on the unit circle.

In the present paper the function-theoretic method has been applied to find the solutions of the flexure problem for three boundaries: (1) inverse of an ellipse with respect to its centre, (2) two orthogonal circular arcs of equal radii and (3) a loop of the lemniscate of Bernoulli. The solutions for the first two boundaries are believed to be new, while that for the third, has been obtained by a different method by Stevenson (1938).

2. Let a pair of rectangular coordinate axes in the plane of the cross-section of the cylinder of length l be taken at its centroid and let the central line of the cylinder be taken as the axis of z, and let P_1 and P_2 be the components of the load parallel to x, y directions respectively. If I_{11}, I_{22}, I_{12}, be the moments and products of inertia of the cross-section about the axes of x and y, and

$$k_1 = (P_1 I_{22} - P_2 I_{11})/(I_{11} I_{22} - I_{12}^2), \quad k_2 = (P_2 I_{11} - P_1 I_{12})/(I_{11} I_{22} - I_{12}^2) \tag{2.1}$$

and $-k_0/(2\mu)$, the twist per unit length of the cylinder, the complex torsion-flexure function is (Ghosh, 1947, p. 12)

$$f = -\tfrac{1}{2} k_0 f_0 + \tfrac{1}{2}(k_1 f_1 + k_2 f_2)/(1+\sigma) \tag{2.2}$$

where f_0 is the complex torsion function for the boundary, and (Ghosh, 1947, p. 10),

$$f_1 = \tfrac{1}{12}(1+2\sigma)z^3 + f_{11} + (1+\sigma)\bar{f}_{12}, \tag{2.3}$$

$$f_2 = \tfrac{1}{12}(1+2\sigma)iz^3 + f_{21} + (1+\sigma)f_{22}, \tag{2.4}$$

(z being equal to $x+iy$).

Let $z = \omega(\zeta)$ be the mapping function which gives the conformal representation of the cross-section of the cylinder on the unit circle ($\zeta \leq 1$) and let γ denote the circumference of the unit circle. Then f_0 is given by (Muschelisvili, 1929),

$$f_0 = \frac{1}{2\pi}\int_\gamma \frac{\omega(\sigma)\bar{\omega}(1/\sigma)}{\sigma-\zeta}\,d\sigma \tag{2.5}$$

and the canonical complex flexure functions f_{11}, f_{12}, f_{21}, f_{22} are given by (Ghosh, 1947, pp. 10-11),

$$f_{11}+if_{21} = -\frac{i}{4\pi}\int_{\gamma} \frac{[\omega(\sigma)]^2\bar{\omega}(1/\sigma)}{\sigma-\zeta}\,d\sigma \tag{2.6}$$

$$f_{11}-if_{21} = \frac{i}{4\pi}\int_{\gamma} \frac{\omega(\sigma)[\bar{\omega}(1/\sigma)]^2}{\sigma-\zeta}\,d\sigma \tag{2.7}$$

$$f_{12}+if_{22} = -\frac{i}{2\pi}\int^{\zeta} d\zeta \int_{\gamma} \frac{[\omega(\sigma)]^2\,\bar{\omega}'(1/\sigma)}{\sigma^2(\sigma-\zeta)}\,d\sigma \tag{2.8}$$

$$f_{12}-if_{22} = -\frac{i}{2\pi}\int^{\zeta} d\zeta \int_{\gamma} \frac{[\bar{\omega}(1/\sigma)]^2\omega'(\sigma)}{\sigma-\zeta}\,d\sigma. \tag{2.9}$$

It may be noted that f_0, f_1, f_2 and f_{11}, f_{21}, f_{12}, f_{22} are all analytic functions of z in the z-plane for the cross-section and are therefore analytic functions of ζ within γ.

The coordinates of the centre of flexure are (Ghosh, 1947, p. 7)

$$x_f = \frac{I_{11}N_2-I_{12}N_1}{2(1+\sigma)(I_{11}I_{22}-I_{12}^2)}, \quad y_f = \frac{I_{12}N_2-I_{22}N_1}{2(1+\sigma)(I_{11}I_{22}-I_{12}^2)} \tag{2.10}$$

where (Ghosh, 1947, p. 12)

$$N_1 = J_{11}+\tfrac{1}{4}(1-2\sigma)J_{12}, \qquad N_2 = J_{21}-\tfrac{1}{4}(1-2\sigma)J_{22} \tag{2.11}$$

$$J_{11} = \tfrac{1}{2}\mathcal{R}\int_{\gamma} \omega(\zeta)\bar{\omega}(1/\zeta)df_{11}(\zeta)+\tfrac{1}{2}(1+\sigma)\mathcal{R}\int_{\gamma}\omega(\zeta)\bar{\omega}(1/\zeta)df_{12}(\zeta) \tag{2.12}$$

$$J_{21} = \tfrac{1}{2}\mathcal{R}\int_{\gamma} \omega(\zeta)\bar{\omega}(1/\zeta)df_{21}(\zeta)+\tfrac{1}{2}(1+\sigma)\mathcal{R}\int_{\gamma}\omega(\zeta)\bar{\omega}(1/\zeta)df_{22}(\zeta) \tag{2.13}$$

$$J_{12}-iJ_{22} = -\tfrac{1}{4}\int_{\gamma}[\omega(\zeta)\bar{\omega}(1/\zeta)]^2 d\omega(\zeta). \tag{2.14}$$

3. We consider an isotropic elastic cylinder the cross-section of which is the inverse of an ellipse with respect to its centre. The centroid being the origin, and the coordinate axes being the principal axes at the centroid, the function which maps the cross-section conformally on the unit circle is (Sokolnikoff, 1946, p. 176)

$$z = \omega(\zeta) = \frac{k\zeta}{\zeta^2+a^2} \quad (a>1,\ k>0) \tag{3.1}$$

The solution of the torsion problem for this boundary is given by Sokolnikoff (1946, p. 178) as,

$$f_0 = \frac{ia^2k^2}{(a^4-1)(\zeta^2+a^2)}. \tag{3.2}$$

From (2.6) and (3.1),
$$f_{11}+if_{21} = \frac{-ik^3}{4\pi a^2}\int_{\gamma}\frac{\sigma^3}{(\sigma^2+1/a^2)(\sigma^2+a^2)^2}\cdot\frac{d\sigma}{\sigma-\zeta},$$

The singularities of the integrand within the unit circle γ are ζ, i/a, $-i/a$ and we have by Cauchy's Theorem of residues,

$$f_{11}+if_{21} = \frac{k^3}{2}\left[\frac{\zeta^3}{(a^2\zeta^2+1)(a^2+\zeta^2)^2}+\frac{a^2\zeta}{(a^4-1)^2(a^2\zeta^2+1)}\right]. \tag{3.3}$$

From (2.7) and (3.1),

$$f_{11}-if_{21} = \frac{ik^3}{4\pi}\int_\gamma \frac{\sigma^3}{(\sigma^2+a^2)(a^2\sigma^2+1)^2}\,\frac{d\sigma}{\sigma-\zeta}.$$

The singularities of the integrand beteen γ and a circle of large radius being ai, and $-ai$ and the integral taken over the large circle being zero, we have

$$f_{11}-if_{21} = \frac{k^3a^2\zeta}{2(a^4-1)^2(\zeta^2+a^2)}. \tag{3.4}$$

Therefore from (3.3) and (3.4) we get,

$$f_{11} = \frac{k^3}{4}\left[\frac{\zeta^3}{(a^2+\zeta^2)^2(a^2\zeta^2+1)}+\frac{a^2(a^2+1)(\zeta^2+1)\zeta}{(a^4-1)^2(a^2+\zeta^2)(a^2\zeta^2+1)}\right] \tag{3.5}$$

$$f_{21} = \frac{-ik^3}{4}\left[\frac{\zeta^3}{(a^2+\zeta^2)^2(a^2\zeta^2+1)}+\frac{a^2(a^2-1)(1-\zeta^2)\zeta}{(a^4-1)^2(a^2+\zeta^2)(a^2\zeta^2+1)}\right] \tag{3.6}$$

We calculate $f_{12}+if_{22}$ from (2.8), and we have,

$$\int_\gamma \frac{[\omega(\sigma)]^2\bar{\omega}'(1/\sigma)}{\sigma^2(\sigma-\zeta)}\,d\sigma = k^3\int_\gamma \frac{\sigma^2(a^2\sigma^2-1)}{(a^2+\sigma^2)^2(a^2\sigma^2+1)^2}\cdot\frac{d\sigma}{\sigma-\zeta}.$$

The singularities of the integrand between the unit circle γ, and a circle of large radius are ai, $-ai$ and the integral taken over the large circle being zero, we get the integral

$$= -k^3.2\pi i[\text{sum of the residues of the integrand at } \sigma = ai, \text{ and } \sigma = -ai]$$

$$= 2\pi ik^3\left[\frac{a^8+6a^4+1}{2(a^4-1)^3(a^2+\zeta^2)}-\frac{(a^4+1)(\zeta^2-a^2)}{2(a^4-1)^2(\zeta^2+a^2)^2}\right].$$

Therefore from (2.8), using this result, we get

$$f_{12}+if_{22} = k^3\left[\frac{a^8+6a^4+1}{2(a^4-1)^3a}\tan^{-1}\frac{\zeta}{a}+\frac{a^4+1}{2(a^4-1)^2}\,\frac{\zeta}{a^2+\zeta^2}\right]. \tag{3.7}$$

We calculate $f_{12}-if_{22}$ from (2.9), and we have

$$\int_\gamma [\bar{\omega}(1/\sigma)]^2\omega'(\sigma)\frac{d\sigma}{\sigma-\zeta} = k^3\int_\gamma \frac{\sigma^2(a^2-\sigma^2)}{(a^2\sigma^2+1)^2(a^2+\sigma^2)^2}\,\frac{d\sigma}{\sigma-\zeta}$$

and proceeding as in the last integral, this is

$$= -2\pi ik^3[\text{sum of the residues of the integrand at } \sigma = ai, \text{ and } \sigma = -ai]$$

$$= -2\pi ik^3\left[\frac{2a^2(a^4+1)}{(a^4-1)^3(\zeta^2+a^2)}+\frac{a^2(a^2-\zeta^2)}{(a^4-1)^2(\zeta^2+a^2)^2}\right].$$

Therefore

$$f_{12}-if_{22} = -k^3\left[\frac{2a(a^4+1)}{(a^4-1)^3}\tan^{-1}\frac{\zeta}{a}+\frac{a^2}{(a^4-1)^2}\frac{\zeta}{\zeta^2+a^2}\right] \tag{3.8}$$

From (3.7) and (3.8) we get

$$f_{12} = \frac{k^3}{4}\left[\frac{a^2-1}{a(a^2+1)^3}\tan^{-1}\frac{\zeta}{a}+\frac{\zeta}{(a^2+1)^2\,(a^2+\zeta^2)}\right], \tag{3.9}$$

$$f_{22} = \frac{-ik^3}{4}\left[\frac{a^2+1}{a(a^2-1)^3}\tan^{-1}\frac{\zeta}{a}+\frac{\zeta}{(a^2-1)^2\,(a^2+\zeta^2)}\right]. \tag{3.10}$$

To complete the solution, we require the values of k_1, k_2. Since, the axes are the principal axes at the centroid, we have from (2.1),

$$k_1 = P_1/I_{11}, \qquad k_2 = P_2/I_{22}.$$

Also we find at once,

$$I_{11} = \frac{\pi k^4}{4(a^4-1)^4}(a^8-2a^6+4a^4-2a^2+1),$$

$$I_{22} = \frac{\pi k^4}{4(a^4-1)^4}(a^8+2a^6+4a^4+2a^2+1),$$

the centre of flexure is obviously at the origin.

4. As a second example, we consider the flexure problem for an isotropic elastic cylinder whose cross-section is bounded by two orthogonal circular arcs of equal radii. The mapping function which makes a conformal representation of such a cross-section on the unit circle is (Sokolnikoff, 1946, p. 181)

$$\zeta = \frac{2z}{z^2+1} \quad \text{or} \quad z = \omega(\zeta) = \frac{1-(1-\zeta^2)^{\frac{1}{2}}}{\zeta}. \tag{4.1}$$

The torsion function is given by Sokolnikoff (p. 181)

$$f_0(\zeta) = -i(1-\zeta^2)^{\frac{1}{2}}-\frac{i(1-\zeta^2)}{\pi\zeta}\log\frac{1-\zeta}{1+\zeta} \tag{4.2}$$

The flexure functions f_{11}, f_{21}, f_{12}, f_{22} are now calculated. From (2.6), (2.7) and (4.1),

$$f_{11}+if_{21} = -\frac{i}{4\pi}\int_\gamma\left[2-\sigma^2-2(1-\sigma^2)^{\frac{1}{2}}\right]\left[1-(1-1/\sigma^2)^{\frac{1}{2}}\right]\frac{d\sigma}{\sigma(\sigma-\zeta)}, \tag{4.3}$$

$$f_{11}-if_{21} = \frac{i}{4\pi}\int_\gamma\left[1-(1-\sigma^2)^{\frac{1}{2}}\right]\left[2-1/\sigma^2-2(1-1/\sigma^2)^{\frac{1}{2}}\right]\frac{\sigma}{\sigma-\zeta}d\sigma. \tag{4.4}$$

we have,

$$\int_\gamma\frac{2-\sigma^2}{\sigma(\sigma-\zeta)}\,d\sigma = -2\pi i\zeta, \tag{4.5}$$

$$\int_\gamma(2-1/\sigma^2)\frac{\sigma}{\sigma-\zeta}\,d\sigma = 4\pi i\zeta. \tag{4.7}$$

To evaluate

$$I_1 = \int_\gamma (1-\sigma^2)^{\frac{1}{2}} \frac{d\sigma}{\sigma(\sigma-\zeta)},$$

$$I_2 = \int_\gamma (1-\sigma^2)^{\frac{1}{2}}(2-1/\sigma^2)\frac{\sigma}{\sigma-\zeta}d\sigma,$$

we perform contour integrations by drawing cuts along the real axis from -1 to $-\infty$, and $+1$ to $+\infty$, and we omit the points $+1$ and -1 by drawing small semi-circles with these points as centres and lying within γ and omit from γ, the parts intercepted by these semi-circles. Therefore, the integrals taken along the remaining part of γ and the two semi-circles can be evaluated by Cauchy's Theorem of residues. It is seen at once that the integrals taken along the semi-circles tend to zero as the radii tend to zero. We find,

$$I_1 = 2\pi i\left[\frac{(1-\zeta^2)^{\frac{1}{2}}}{\zeta} - \frac{1}{\zeta}\right], \tag{4.6}$$

$$I_2 = -2\pi i\left[(1-\zeta^2)^{\frac{1}{2}}\frac{1-2\zeta^2}{\zeta} - \frac{1}{\zeta}\right]. \tag{4.8}$$

To evaluate

$$I_3 = \int_\gamma \frac{2-\sigma^2}{\sigma(\sigma-\zeta)}(1-1/\sigma^2)^{\frac{1}{2}}d\sigma,$$

$$I_4 = \int_\gamma (1-1/\sigma^2)^{\frac{1}{2}}\frac{\sigma}{\sigma-\zeta}d\sigma,$$

we draw cuts along the real axis from -1 to $+1$ and omit the points $+1$ and -1 by drawing small semi-circles with these points as centre and lying outside γ. The integrals taken along the remaining part of γ and two semi-circles are respectively equal to integrals taken along a circle Σ of very large radius. The integrals taken along the semi-circles tend to zero as the radii tend to zero, and making the radius of Σ tend to ∞ we get

$$I_3 = -2\pi i\zeta, \tag{4.9}$$

$$I_4 = 2\pi i\zeta. \tag{4.10}$$

To evaluate

$$I_5 = \int_\gamma \frac{(1-\sigma^2)^{\frac{1}{2}}(1-1/\sigma^2)^{\frac{1}{2}}}{\sigma(\sigma-\zeta)}d\sigma,$$

$$I_6 = \int_\gamma (1-\sigma^2)^{\frac{1}{2}}(1-1/\sigma^2)^{\frac{1}{2}}\frac{\sigma}{\sigma-\zeta}d\sigma,$$

we draw a cut along the real axis from $-\infty$ to $+\infty$ and evaluate the integrals separately for the upper and lower semi-circles noting that the integrand changes sign as we cross the real axis. We get,

$$I_5 = 2i\left[-\frac{2}{\zeta} + \frac{1-\zeta^2}{\zeta^2}\log\frac{1+\zeta}{1-\zeta}\right], \tag{4.11}$$

$$I_6 = 2i\left[2\zeta + (1-\zeta^2)\log\frac{1+\zeta}{1-\zeta}\right]. \tag{4.12}$$

Hence from (4.3) using (4.5), (4.7), (4.9) and (4.11) and from (4.4) using (4.6), (4.8), (4.10) and (4.12) we have

$$f_{11}+if_{21} = -\frac{(1-\zeta^2)^{\frac{1}{2}}}{\zeta}+\frac{1}{\zeta}-\frac{2}{\pi\zeta}+\frac{1-\zeta^2}{\pi\zeta^2}\log\frac{1+\zeta}{1-\zeta},$$

$$f_{11}-if_{21} = -\frac{1}{2}\left[(1-\zeta^2)^{\frac{1}{2}}\frac{(1-2\zeta^2)}{\zeta}-\frac{1}{\zeta}+\frac{4\zeta}{\pi}+\frac{2(1-\zeta^2)}{\pi}\log\frac{1+\zeta}{1-\zeta}\right].$$

Therefore

$$f_{11} = \frac{1}{4}\left[-(1-\zeta^2)^{\frac{1}{2}}\frac{3-2\zeta^2}{\zeta}+\frac{3}{\zeta}-\frac{4}{\pi\zeta}(1+\zeta^2)+\frac{2}{\pi\zeta^4}(1-\zeta^2)^2\log\frac{1+\zeta}{1-\zeta}\right], \tag{4.13}$$

$$f_{21} = \frac{1}{4}i\left[(1-\zeta^2)^{\frac{1}{2}}\frac{(1+2\zeta^2)}{\zeta}-\frac{1}{\zeta}+\frac{4}{\pi}\frac{1-\zeta^2}{\zeta}-\frac{2}{\pi}\frac{1-\zeta^4}{\zeta^2}\log\frac{1+\zeta}{1-\zeta}\right]. \tag{4.14}$$

We calculate $f_{12}+if_{22}$ and $f_{12}-if_{22}$ from (2.8) and (2.9) and proceeding as before, we get

$$\int_{\gamma}\frac{[\omega(\sigma)]^2\omega'(1/\sigma)}{\sigma^2(\sigma-\zeta)}d\sigma = 4i\left[\frac{1}{\zeta}\log\frac{1+\zeta}{1-\zeta}+\pi\left\{\frac{(1-\zeta^2)^{\frac{1}{2}}}{\zeta^2}-\frac{1}{\zeta^2}\right\}\right],$$

$$\int_{\gamma}[\bar{\omega}(1/\sigma)]^2\omega'(\sigma)\frac{d\sigma}{\sigma-\zeta} = 2\pi i\left[\frac{1}{\zeta^2}+\frac{2\zeta^2-1}{\zeta^2(1-\zeta^2)^{\frac{1}{2}}}-\frac{2}{\pi\zeta}\log\frac{1+\zeta}{1-\zeta}\right].$$

Therefore

$$f_{12}+if_{22} = -\frac{2}{\zeta}(1-\zeta^2)^{\frac{1}{2}}-2\sin^{-1}\zeta+\frac{2}{\zeta}+\frac{4}{\pi}\left(\zeta+\frac{\zeta^3}{3^2}+\frac{\zeta^5}{5^2}+\frac{\zeta^7}{7^2}+\cdots\right),$$

$$f_{12}-if_{22} = -\frac{1}{\zeta}+\frac{(1-\zeta^2)^{\frac{1}{2}}}{\zeta}+2\sin^{-1}\zeta-\frac{4}{\pi}\left(\zeta+\frac{\zeta^3}{3^2}+\frac{\zeta^5}{5^2}+\frac{\zeta^7}{7^2}+\cdots\right).$$

Therefore

$$f_{12} = \frac{1}{2}\left[\frac{1}{\zeta}-\frac{(1-\zeta^2)^{\frac{1}{2}}}{\zeta}\right], \tag{4.15}$$

$$f_{22} = \frac{1}{2}i\left[\frac{3(1-\zeta^2)^{\frac{1}{2}}}{\zeta}-\frac{3}{\zeta}+4\sin^{-1}\zeta-\frac{8}{\pi}\left(\zeta+\frac{\zeta^7}{3^2}+\frac{\zeta^5}{5^2}+\frac{\zeta^7}{7^2}+\cdots\right)\right]. \tag{4.16}$$

Also, we get at once, $k_1 = P_1/I_{11}$ and $k_2 = P_2/I_{22}$, where

$$I_{11} = \frac{\pi}{2}-\frac{4}{3} \quad \text{and} \quad I_{22} = \frac{3\pi}{2}-\frac{14}{3}.$$

The centre of flexure is at the centroid. The flexure problem for the boundary is therefore solved completely.

5. As a third example, we solve the flexure problem for a cylinder whose cross-section is a loop of the lemniscate of Bernoulli. Taking the origin at the centroid, the corresponding mapping function is

$$z = \omega(\zeta) = a(1+\zeta)^{\frac{1}{2}}-\pi a/4. \tag{5.1}$$

The torsion function for this is given by (Sokolnikoff, 1946, p. 179)

$$f_0(\zeta) = \frac{a^2}{\pi}\frac{1+\zeta}{\sqrt{\zeta}}\log\frac{1+i\sqrt{\zeta}}{1-i\sqrt{\zeta}}. \tag{5.2}$$

From (2.6), (2.7) and (5.1), we get

$$f_{11}+if_{21} = -\frac{ia'}{4\pi}\int_\gamma \left\{1+\sigma+\frac{\pi^2}{16}-\frac{\pi}{2}(1+\sigma)^{\frac12}\right\}\left\{(1+1/\sigma)^{\frac12}-\frac{\pi}{4}\right\}\frac{d\sigma}{\sigma-\zeta}, \tag{5.3}$$

$$f_{11}-if_{21} = \frac{ia'}{4\pi}\int_\gamma \left\{(1+\sigma)^{\frac12}-\frac{\pi}{4}\right\}\left\{1+\frac{1}{\sigma}+\frac{\pi^2}{16}-\frac{\pi}{2}(1+1/\sigma)^{\frac12}\right\}\frac{d\sigma}{\sigma-\zeta}. \tag{5.4}$$

We have, omitting additive constants,

$$\int_\gamma \left(1+\sigma+\frac{\pi^2}{16}\right)\frac{d\sigma}{\sigma-\zeta} = 2\pi i\zeta, \tag{5.5}$$

$$\int_\gamma \left(1+\frac{1}{\sigma}+\frac{\pi^2}{16}\right)\frac{d\sigma}{\sigma-\zeta} = \text{constant}. \tag{5.6}$$

To evaluate

$$I_1' = \int_\gamma (1+\sigma)^{\frac12}\frac{d\sigma}{\sigma-\zeta},$$

$$I_2' = \int_\gamma (1+\sigma)^{\frac12}\left(1+\frac{1}{\sigma}+\frac{\pi^2}{16}\right)\frac{d\sigma}{\sigma-\zeta},$$

we draw a cut along the real axis from -1 to $-\infty$ and omit the point -1 by describing a small semi-circle with -1 as centre and lying within γ. The integrals along the remaining part of γ and the small semi-circle can be evaluated by Cauchy's Theorem of residues. It can be shown that the integrals taken along the small semi-circle tend to zero, as the radius tends to zero. Therefore

$$I_1' = 2\pi i(1+\zeta)^{\frac12}, \tag{5.7}$$

$$I_2' = 2\pi i\left[\frac{(1+\sigma)^{3/2}}{\zeta}+\frac{\pi^2}{16}(1+\zeta)^{\frac12}-\frac{1}{\zeta}\right]. \tag{5.8}$$

To evaluate

$$I_3' = \int_\gamma \left(1+\sigma+\frac{\pi^2}{16}\right)\left(1+\frac{1}{\sigma}\right)^{\frac12}\frac{d\sigma}{\sigma-\zeta},$$

$$I_4' = \int_\gamma \left(1+\frac{1}{\sigma}\right)^{\frac12}\frac{d\sigma}{\sigma-\zeta},$$

we draw a cut along the real axis from -1 to 0 and omit the point -1, by describing a small semi-circle with -1 as centre and lying outside γ. The integrals taken along the remaining part of γ and the semi-circle can be taken along a circle Σ of large radius. The integrals taken along the small semi-circle tend to zero, as the radius tends to zero, and making the radius of Σ tend to ∞, we get omitting the constant,

$$I_3' = 2\pi i\zeta, \tag{5.9}$$

$$I_4' = \text{constant}. \tag{5.10}$$

To evaluate

$$I_5' = \int_\gamma (1+\sigma)^{\frac12}(1+1/\sigma)^{\frac12}\frac{d\sigma}{\sigma-\zeta},$$

we draw a cut along the real axis from $-\infty$ to 0 and evaluate the integral along γ from $\sigma = e^{-i\pi}$ to $\sigma = e^{i\pi}$, and we get,

$$I_5' = \frac{2}{\sqrt{\zeta}}(1+\zeta)\log\frac{1+i\sqrt{\zeta}}{1-i\sqrt{\zeta}} + \text{constant} \tag{5.11}$$

Hence from (5.3) using (5.5), (5.7), (5.9) and (5.11), and from (5.4) using (5.6), (5.8), (5.10) and (5.11) we get, omitting constants,

$$f_{11}+if_{21} = \frac{a^3}{4}\left[\left(2-\frac{\pi}{2}\right)\zeta + \frac{i}{\sqrt{\zeta}}(1+\zeta)\log\frac{1+i\sqrt{\zeta}}{1-i\sqrt{\zeta}} + \frac{\pi^2}{4}(1+\zeta)^{\frac{1}{2}}\right],$$

$$f_{11}-if_{21} = -\frac{a^3}{4}\left[\frac{2(1+\zeta)^{3/2}}{\zeta} + \frac{\pi^2}{8}(1+\zeta)^{\frac{1}{2}} - \frac{2}{\zeta} + \frac{i}{\sqrt{\zeta}}(1+\zeta)\log\frac{1+i\sqrt{\zeta}}{1-i\sqrt{\zeta}}\right]$$

Therefore

$$f_{11} = \frac{a^3}{8}\left[\left(2-\frac{\pi}{2}\right)\zeta + \frac{\pi^2}{8}(1+\zeta)^{\frac{1}{2}} - \frac{2(1+\zeta)^{3/2}}{\zeta} + \frac{2}{\zeta}\right], \tag{5.12}$$

$$f_{21} = -\frac{ia^3}{8}\left[\left(2-\frac{\pi}{2}\right)\zeta + \frac{2i}{\sqrt{\zeta}}(1+\zeta)\log\frac{1+i\sqrt{\zeta}}{1-i\sqrt{\zeta}} + \frac{3\pi^2}{8}(1+\zeta)^{\frac{1}{2}} - \frac{2}{\zeta} + \frac{2(1+\zeta)^{3/2}}{\zeta}\right] \tag{5.13}$$

We calculate $f_{12}+if_{22}$ and $f_{12}-if_{22}$ from (2.8) and (2.9), and proceeding, as before, we get

$$\int_\gamma \frac{[\omega(\sigma)]^2\omega'(1/\sigma)d\sigma}{\sigma^2(\sigma-\zeta)} = \pi i a^3\left[\frac{1}{\zeta} + \frac{i}{2\zeta\sqrt{\zeta}}\log\frac{1+i\sqrt{\zeta}}{1-i\sqrt{\zeta}}\right],$$

$$\int_\gamma [\bar{\omega}(1/\sigma)]^2\omega'(\sigma)\frac{d\sigma}{\sigma-\zeta} = \pi i a^3\left[\left(1+\frac{1}{\zeta}+\frac{\pi^2}{16}\right)\frac{1}{(1+\zeta)^{\frac{1}{2}}} - \frac{1}{\zeta} + \frac{i}{2\sqrt{\zeta}}\log\frac{1+i\sqrt{\zeta}}{1-i\sqrt{\zeta}}\right].$$

Therefore

$$f_{12}+if_{22} = \frac{a^2}{2}\left[\log(1+\zeta) - \frac{i}{\sqrt{\zeta}}\log\frac{1+i\sqrt{\zeta}}{1-i\sqrt{\zeta}}\right],$$

$$f_{12}-if_{22} = \frac{a^3}{2}\left[\frac{2}{3}(1+\zeta)^{3/2} + \frac{\pi^2}{8}(1+\zeta)^{\frac{1}{2}} + \log\frac{1+\zeta}{\zeta} + i\sqrt{\zeta}\log\frac{1+i\sqrt{\zeta}}{1-i\sqrt{\zeta}}\right],$$

Therefore

$$f_{12} = \frac{a^3}{4}\left[\frac{2}{3}(1+\zeta)^{3/2} + \frac{\pi^2}{8}(1+\zeta)^{\frac{1}{2}} - \frac{i}{\sqrt{\zeta}}(1-\zeta)\log\frac{1+i\sqrt{\zeta}}{1-i\sqrt{\zeta}} + \log\frac{(1+\zeta)^2}{\zeta}\right], \tag{5.14}$$

$$f_{22} = \frac{a^3 i}{4}\left[\frac{2}{3}(1+\zeta)^{3/2} + \frac{\pi^2}{8}(1+\zeta)^{\frac{1}{2}} - \log\zeta + \frac{i(1+\zeta)}{\sqrt{\zeta}}\log\frac{1+i\sqrt{\zeta}}{1-i\sqrt{\zeta}}\right]. \tag{5.15}$$

It remains to find the coordinates of the centre of flexure

$$I_{11} = \frac{a^4}{8}\left[\frac{8}{3}+\pi-\frac{\pi^2}{2}\right], \qquad I_{22} = \frac{a^4}{8}\left[\pi-\frac{8}{3}\right].$$

We find at once,

$$k_1 = P_1/I_{11} \text{ and } k_2 = P_2/I_{22}.$$

The coordinates of the centre of flexure are given by

$$x_f = \frac{N_2}{2(1+\sigma)I_{22}}, \quad y_f = 0.$$

We calculate $J_{12}-iJ_{22}$ and J_{21} from (2.14) and (2.13) and we get

$$J_{12}-iJ_{22} = -\frac{ia^5}{8}\left[\frac{2\pi}{3}-\pi^2+\frac{\pi^3}{4}\right];$$

and to evaluate

$$\int_\gamma \omega(\zeta)\bar{\omega}(1/\zeta)df_{21}(\zeta), \quad \int_\gamma \omega(\zeta)\bar{\omega}(1/\zeta)df_{22}(\zeta), \tag{5.16}$$

we use the following results:

$$\int_\gamma\left\{\frac{\zeta-2}{2\zeta^2}(1+\zeta)-\frac{\pi}{8}\left(2-\frac{5\pi}{4}+\frac{2}{\zeta^2}\right)+(1+\zeta)^{\frac{1}{2}}\left(\frac{4-\pi}{4}+\frac{1}{\zeta^2}-\frac{1}{\zeta}-\frac{\pi}{8}.\frac{\zeta-2}{\zeta^2}-\frac{3\pi^3}{128(1+\zeta)}\right)\right\}d\zeta = -2\pi i,$$

$$\int_\gamma\left[\frac{1}{2}\left(1+\frac{1}{\zeta}+\frac{\pi^2}{16}+\frac{\pi}{4\zeta}\right)-(1+\zeta)^{\frac{1}{2}}\left\{\frac{1}{\zeta}+\frac{\pi}{8}\left(1+\frac{\pi^2}{16}+\frac{1}{\zeta}\right)\frac{1}{1+\zeta}\right\}\right]d\zeta = -\pi i,$$

$$\int_\gamma\left\{\frac{(1+\zeta)(\zeta-2)}{\zeta^3}+\frac{3\pi^2}{16}-\frac{\pi}{4}\left(2-\frac{\pi}{2}\right)-\frac{\pi}{2\zeta^2}+\frac{\pi}{2\zeta}\right\}(1+1/\zeta)^{\frac{1}{2}}d\zeta = 2\pi i\left(-\frac{1}{2}+\frac{5\pi^2}{32}+\frac{\pi}{4}\right),$$

$$\int_\gamma\left\{1+\frac{1}{\zeta}+\frac{\pi^2}{16}+\frac{\pi}{2\zeta}\right\}(1+1/\zeta)^{\frac{1}{2}}d\zeta = i\left(3\pi+\pi^2+\frac{\pi^3}{16}\right),$$

$$\int_\gamma\left\{2-\frac{\pi}{2}+\frac{2}{\zeta^2}-\frac{2}{\zeta}-\frac{\pi}{4}.\frac{\zeta-2}{\zeta^2}-\frac{3\pi^3}{64(1+\zeta)}\right\}(1+\zeta)^{\frac{1}{2}}(1+1/\zeta)^{\frac{1}{2}}d\zeta = -i\left(\frac{16}{3}+2\pi+\frac{3\pi^3}{16}\right),$$

$$\int_\gamma\left\{\frac{2}{\zeta}+\frac{\pi}{4(1+\zeta)}\left(1+1/\zeta+\frac{\pi^2}{16}\right)\right\}(1+\zeta)^{\frac{1}{2}}(1+1/\zeta)^{\frac{1}{2}}d\zeta = i\left(16+2\pi+\frac{\pi^3}{16}\right),$$

$$\int_\gamma (1+\zeta)^{\frac{1}{2}}\frac{1-\zeta}{\zeta\sqrt{\zeta}}\log\frac{1+i\sqrt{\zeta}}{1-i\sqrt{\zeta}}d\zeta = -4\pi,$$

$$\int_\gamma \frac{1-\zeta}{\zeta\sqrt{\zeta}}(1+1/\zeta)^{\frac{1}{2}}\log\frac{1+i\sqrt{\zeta}}{1-i\sqrt{\zeta}}d\zeta = \pi(\pi-4),$$

$$\int_\gamma \frac{1-\zeta}{\zeta\sqrt{\zeta}}(1+\zeta)^{\frac{1}{2}}(1+1/\zeta)^{\frac{1}{2}}\log\frac{1+i\sqrt{\zeta}}{1-i\sqrt{\zeta}}d\zeta = -8,$$

$$\int_\gamma\left\{2/\zeta+\frac{i(1-\zeta)}{\zeta\sqrt{\zeta}}\log\frac{1+i\sqrt{\zeta}}{1-i\sqrt{\zeta}}\right\}d\zeta = 0.$$

From (2.13) and substituting these values in (5.16) we get

$$J_{21} = \frac{a^5}{16}\left[\left(\frac{80}{3}-5\pi-\frac{3\pi^2}{2}+\frac{\pi^3}{8}\right)+\sigma\left(24-2\pi-\pi^2-\frac{\pi^3}{4}\right)\right].$$

From above and (2.11) we have,

$$N_2 = \frac{a^5}{16}\left(\left[\frac{80}{3}-\frac{16\pi}{3}-\pi^2\right)+\sigma\left(24-\frac{4\pi}{3}-2\pi^2\right)\right].$$

Therefore

$$x_f = \frac{a}{(3\pi-8)(1+\sigma)}\left[(20-4\pi-\tfrac{3}{4}\pi^2)+\sigma(18-\pi-\tfrac{3}{2}\pi^2)\right]. \tag{5.17}$$

If we replace σ by $\sigma/(1-\sigma)$, the new σ being the modified Poisson's ratio of the generalised plane stress, the above result (5.17) is in agreement with Stevenson's result worked out by a different method.

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QUASI-GROUPS AND NONASSOCIATIVE SYSTEMS—I

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1. Introduction

Sometimes ago, it was shown by the author that a 3-web can be represented algebraically by means of a quasigroups. Since then, in a series of remarkable papers, Albert (1943, 1944), Bruck (1944, 1946) and others have made valuable contributions to the theory of quasi-groups. As this subject holds a key position in the theory of general nonassociative systems, it is now possible to unearth their mysteries to some extent. With this object in view, quasigroups and nonassociative systems will be studied in these papers in some detail.

A system of elements a, b, c etc. is called a quasi-group which is such that the third element of the relation $ab = c$ is uniquely defined by the other two elements. This means that the system is closed with respect to the operation of multiplication and the equations $ax = b$ and $ya = b$ are uniquely solvable for every a and b. In particular, the equations $ax = a$ and $ya = a$ are uniquely solvable for each element a. The solution of the equation $ax = a$ is called the right identity of a and will be denoted by $e_{r,a}$. Similarly the left identity $e_{l,a}$ of a is the solution of the equation $ya = a$. Hence in a quasi-group there exist a unique right identity and a unique left identity of each element. These identity elements are in general different for different elements and even for any given element, the left identity is different from the right identity. This may be easily be seen from the example:

	a	b	c	d
a	d	a	c	b
b	c	d	b	a
c	a	b	d	c
d	b	c	a	d

Here $a(cd) = ac = c$ and $(ac)d = cd = c$. Thus $a(cd) = (ac)d$. But $a(bd) = a.a = d$, $(ab)d = ad = b$, *i.e.*, $a(bd) \neq (ab)d$. So the associative law does not hold in general. The right identity of a is b and the left identity of a is c.

As $(e_{l,a}a)e_{r,a} = a = e_{l,a}(ae_{r,a})$, an element a is central associative to its identities.

On the other hand, left identities (as well as right identities) of distinct elements are distinct, unless the associative-commutative law holds.

For, let a and b have the same left identity e. Then

$$eb = b \text{ and } ea = a.$$

Take an element c such that $ca = b$. Then

$$eb = e(ca) \text{ and } b = ca = c(ea);$$

so

$$e(ca) = c(ea).$$

Thus if the left identity e of a is also the left identity of another element, then there exist a certain element c associative-commutative to e and a.

It is easy to construct quasi-groups in which all elements have the same left identity. For example, in the quasi-group whose multiplication table is

	a	b	c
a	a	b	c
b	c	a	b
c	b	c	a

a is the common left identity and there is no common right identity. Such a quasi-group will be called a left loop. On the other hand, if the quasi-group has a unique right identity for all elements, then it will be called a right loop If a quasi-group is both a left and a right loop, then it will be called a loop. This name loop is due to A. A. Albert.

In a loop the identity element is both sided. For if e_1 is the left identity and e_2 is the right identity, then

$$e_1e_2 = e_1 \text{ as } e_2 \text{ is the right identity}$$

and

$$e_1e_2 = e_2 \text{ as } e_1 \text{ is the left identity.}$$

Thus

$$e_1 = e_2.$$

An infinite right loop will now be constructed. The method of construction may be called "the method of distortion." In an infinite additive group G, define a product $a^*b = a-b$ where a, b are elements of G and the difference $a-b$ is the inverse operation of the sum defined in G. For this multiplication

$$a^*b = a-b, \quad b^*a = b-a$$

and

$$(a^*b)^*c = a-b-c, \quad a^*(b^*c) = a-b+c.$$

So

$$a^*b \neq b^*a \text{ and } (a^*b)^*c \neq a^*(b^*c)$$

and the multiplication is non commutative and non-associative. The equation $a^*x = b$ is solvable uniquely, the solution being $x = a^*b$. Similarly the equation $x^*a = b$ is also solvable uniquely; but the solution here is $a+b$ which is the sum defined in the group. Thus the set of all elements of G form a quasi-group with respect to this multiplication.

As $a-0 = a$ in G, $a^*0 = a$ for all a and hence 0 is the unique right identity for all elements of the quasi-group. But $0-a = -a \neq a$ in G; hence $0^*a \neq a$.

So 0 is not the left identity of a.

Thus the group G become distorted into a right loop.

If a quasi-group is associative, then it is a group and hence it has a unique both sided identity element. If weaker associative laws or associative-commutative laws are assumed then the quasi-group may or may not have a unique identity element. For example, consider a quasi-group in which the relation $a(bc) = (ac)b$ holds between any three elements. Then, let a_0 be a fixed element. There exists a unique solution x_0 such that

$$x_0a_0 = a_0.$$

Multiplying with y,

$$y(x_0a_0) = ya_0$$

But

$$y(x_0a_0) = (ya_0)x_0 \text{ and hence } (ya_0)x_0 = ya_0.$$

If a is any other element, then it is possible to choose y such that $ya_0 = a$; so that the above relation gives that $ax_0 = a$. Thus the left identity of one element a_0 is the right identity for all elements a including a_0 itself, *i.e.*,

$$e_{1,a_0} = e_{r,a}.$$

Similarly taking any other element a_1, $e_{1,a_1} = e_{r,a}$, so that

$$e_{1,a_1} = e_{1,a_0}.$$

Thus all left identities are identical and are also identical with the right identity. Hence there exists a common both-sided identity element e in the quasi-group.

Now put $a = e$ in the relation $a(bc) = (ac)b$. Then it reduces to $bc = cb$ and further $a(bc) = a(cb) = (ab)c$. The associative-commutative law breaks up into two laws: the associative law and the commutative law. The quasi-group is therefore an Abelian group.

Again consider a quasi-group in which the Bol-Moufang law $(b(cb))a = b(c(ba))$ holds. Then let a be any given element and p any other arbitrary element of the quasi-group. Then there exists a unique element y such that $ay = p$, a unique element z such that $za = y$ and a unique element x such that $ax = a$. Thus x is the right identity of a. Therefore $a(za) = p$ and hence $(a(za))x = px$. But

$$(a(za))x = a(z(ax)) = a(za) = ay = p$$

and hence $px = p$ which proves that x is the right identity of both p and a. As p is an arbitrary element, x is the common right identity of all elements of the quasi-group.

Thus a quasi-group in which the Bol-Moufang law holds is a right loop.

An interesting remark can be made here. A quasi-group in which the relation that if $ab = cd$, then $ba = dc$ holds, is called a semi-Abelian quasi-group. The relation that if $ab = cd$, then $ba = dc$ was deduced from a closed figure called Dibbert's figure in a 3-web (Choudhury, 1942, p. 93). If a quasi-group is a semi-Abelian loop, then it is Abelian. For if e is the both sided identity, $ab = p = pe$ and hence $ba = ep$. But as e is the both sided identity $ep = p$ and hence $ab = ba$.

The relation may be called the semi-commutativity relation. Thus semi-commutativity in a loop (or a group) gives nothing new beyond commutativity; but in arbitrary quasi-groups the law behaves differently. It is easy to construct an example of such a quasi-group. For example,

	a	b	c	d
a	c	b	d	a
b	b	c	a	d
c	a	d	c	b
d	d	a	b	c

The relations here are

$$(1)\quad ab = ba = cd = dc = b$$

$$(2)\quad ac = bd = cb = da = d$$

and

$$ca = db = bc = ad = a$$

$$(3)\quad a^2 = b^2 = c^2 = d^2 = c.$$

The commutative law signifies that the multiplication table of the system is symmetric about the principal diagonal. The semi-commutativity has a similar, though more complicated, significance. The principal diagonal divides the table into two triangles: the upper triangle and the lower triangle. The semi-commutativity implies that if two elements in the upper triangle (or in the lower triangle) are identical, then the two elements in the other triangle which are symmetric to these elements relative to the principal diagonal must also be identical, and further that if an element in the upper triangle is identical with an element in the lower triangle, then the elements in the symmetrically opposite places should be identical also.

One more remark will be made here. Each given element has four identity elements and so there exist four inverse elements of the given element. Thus the inverse elements of a are the solutions of the equations

1. $ax_1 = e_{r,a}$ 2. $ax_2 = e_{l,a}$

3. $y_1a = e_{r,a}$ 4. $y_2a = e_{l,a}$.

If $e_{r,a} = e_{l,a}$, then $x_1 = x_3$ and $y_1 = y_2$.

2. Homotopy

In the previous article, a right loop has been constructed by the distortion of a group. This method of distortion is very useful in the construction of quasi-groups and indeed it can be generalised so as to get the more general notions of isotopy and homotopy first disscussed by A. A. Albert.

First of all, let us use more general distortions to construct right loops and left loops from a given quasi-group. Let P denote an one to one reversible mapping: $a \to a_1 = P(a)$ a given quasi-group Q onto itself. Further, to every fixed element b of Q there exist two one to one reversible mappings of Q onto itself, namely a right mapping $R_b: x \to xb$ and a left mapping $L_b: x \to bx$. Then $R_b^{-1}(xb) = x$ and $L_b^{-1}(bx) = x$. Now distort the quasi-group Q by the new rule of multiplication

$$a*b = P^{-1}(P(a)L_f^{-1}P(b))$$

where f is any given element of the quasi group Q. Now

$$L_f^{-1}P(b) = L_f^{-1}(b_1) = m_1 \text{ if } b_1 = fm_1.$$

Thus

$$a^*b = P^{-1}(P(a)L_f^{-1}P(b)) = P^{-1}(a_1m_1) = P^{-1}(b_1) \text{ if } b_1 = a_1m_1$$
$$= b$$

Denote by Q^* the groupoid obtained from Q by this distortion. It will now be proved that Q^* is a left loop. Firstly the equation $a^*x = 1$ is uniquely solvable. For, as a and 1 are given, a_1 and 1_1 are known. Then m_1 is determined from the relation $a_1m_1 = 1_1$. Thus $b_1 = fm_1$ is uniquely determined and hence also $x = P^{-1}(b_1) = b$. Next in the equation $x^*b = 1$, b and 1 determine b_1 and 1_1. Then m_1 is given by $b_1 = fm_1$ and a_1 is the solution of $ym_1 = 1_1$. Finally $x = P^{-1}(a_1) = a$. Lastly if e is an element such that $P(e) = e_1 = f$, then e will be the common left identity. For,

$$\begin{aligned} e^*b &= P^{-1}(P(e)L_f^{-1}P(b)) \\ &= P^{-1}(fm_1) = P^{-1}(b_1) = b. \end{aligned}$$

But e is not the right identity. For,

$$\begin{aligned} a^*e &= P^{-1}(P(a)L_f^{-1}P(e)) \\ &= P^{-1}(a_1L_f^{-1}f) \\ &= P^{-1}(a_1k_1), \text{ where } f = fk_1 \\ &\neq P^{-1}(a_1) = a, \text{ unless } a_1 = a_1k_1. \end{aligned}$$

But k_1 can not be the right identity of every a_1 unless the quasi-group Q is a right loop. Thus Q^* is a left loop.

In a similar manner it can be proved that the quasi-group Q can be distorted into a right loop by means of the distortion

$$a^*b = P^{-1}(R_g^{-1}P(a)P(b)).$$

where g is a given element, and into a loop by means of the distortion

$$a^*b = P^{-1}(R_g^{-1}P(a)L_f^{-1}P(b)).$$

The last result is due to A. A. Albert (1943). Here the element e where $P(e) = fg$ is the common both sided identity of the loop.

The above distortions are all special cases of the general distortion

$$a^*b = \sigma_3[\sigma_1(a)\sigma_2(b)]$$

where σ_1, σ_2, σ_3 are three given nonsingular mapping of Q on itself. The distortions of this type are called isotopy and the quasi-group Q^* is called isotopic to Q. Albert's result is therefore that there exists a suitable isotopy distorting a given quasi-group into a loop. On the strength of this theorem, Albert restricts his discussions mainly to loops; but many interesting properties are obscured by this distortion. One such property is the relation of identity elements.

A more general notion is that of homotopy. Two quasi-groups Q and Q^* are connected by a relation of homotopy if there exist two transformations σ_1, σ_2 of Q and a transformation σ_3 of Q^* such that

$$\sigma_3(a^*b) = \sigma_1(a)\sigma_2(b).$$

Q^* will be called homotopic to Q. If σ_1, σ_2 are nonsingular transformations, then the homotopy will be called a simple homotopy. Homomorphism is a particular case of homotopy; for, if $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$, then

$$\sigma(a^*b) = \sigma(a)\sigma(b).$$

Further, if σ_1, σ_2, σ_3 are nonsingular transformations, then

$$a^*b = \sigma_3^{-1}(\sigma_1(a)\sigma_2(b))$$

and hence the homotopy reduces to isotopy.

In the particular case when $\sigma_1 = \tau_1\sigma_3\alpha_1$, $\sigma_2 = \tau_2\sigma_3\alpha_2$, τ_i, α_i being nonsingular, then an isotope of Q is homomorphic to an isotope of Q^*. For,

$$\begin{aligned}\sigma_1(a)\sigma_2(b) &= \tau_1\sigma_3\alpha_1(a)\tau_2\sigma_3\alpha_2(b)\\ &= \sigma_3(a_1)\cdot\sigma_3(b_1)\end{aligned}$$

where we define a new multiplication $x.y$ by $x.y = \tau_1(x)\tau_2(y)$ and put $a_1 = \alpha_1(a)$, $b_1 = \alpha_2(b)$. This new multiplication distort Q into another quasi-group Q_1. Again as $a = \alpha_1^{-1}(a_1)$ $b = \alpha_2^{-1}(b_1)$,

$$\sigma_3(a^*b) = \sigma_3(\alpha_1(a_1)^*\alpha_2^{-1}(b_1)) = \sigma_3(a_1 \times b_1)$$

where another new multiplication

$$\alpha_1^{-1}(a_1)^*\alpha_2^{-1}(b_1) = a_1 \times b_1$$

is defined distorting Q^* into Q_1^*. Thus

$$\sigma_3(a_1 \times b_1) = \sigma_3(a_1).\sigma_3(b_1).$$

This proves that Q_1 is homomorphic to Q_1^*.

The isotopy $a^*b = \sigma_1(a)\sigma_2(b)$ is called a principal isotopy. Q_1 is a principal isotope of Q and Q_1^* is that of Q^*. Homotopy in this case is equivalent to a homomorphism except for principal isotopies.

Again, a simple homotopy can be resolved into two simpler homotopies. Define a new multiplication $a \times b = \sigma_1(a)\sigma_2(b)$ in Q. This new multiplication will distort Q into a principal isotope Q_1 of Q. The relation $\sigma_3(a^*b) = \sigma_1(a)\sigma_2(b)$ then reduces to $\sigma_3(a^*b) = a \times \mathrm{b}$ which is also a special case of the general homotopy obtained by putting $\sigma_1 = \sigma_2 = 1$. This homotopy may be called a shear (or twist). Thus a simple homotopy can be expressed as a product of a principal isotopy and a shear.

A shear homotopy $\sigma(a^*b) = ab$ is an isotopy if σ is a reversible transformation. A shear which preverses the identity relation for every elements must be the identical shear. For, if the relations $ab = a$ and $a^*b = a$ hold at the same time, then $\sigma(a) = a$ for every element a. Hence σ. is the identical transformation. A non-identical shear has therefore the property that there exist at least one element a whose identity element is changed, *i.e.*, if $a^*b = a$, then $ab = \sigma(a) \neq a$. Some more examples of quasi-groups obtained by distortion will be considered now. The following example is due to Albert.

Let A, B, C denote matrices and apply on the ring of matrices the isotopic distortion $A*B = \sigma_1(A)\sigma_2(B)$ where $\sigma_1(A) = A$, $\sigma_2(B) = B^t$, B^t being the transposed matrix of B. Then $A*B$ is a new matrix obtained from A and B by forming scalar products of row vectors of A with row vectors of B. The semigroup of the matrix ring is distorted into a quasi-group. As $E^t = E$, the unit matrix E remain the common right identity; but it is no longer the common left identity. For, $E*A = EA^t = A^t \neq A$. In a similar manner one can get from the distortion of the matrix ring quasi-groups in which the matrix product is formed by the rule "column by row" or "column by column."

Certain other important types of quasi-group are obtained from the Galois fields GF_{p^n} by homotopic distortions. Define a new multiplication in the Galois field by the distortion

$$a*b = a^k b^s$$

so that $\sigma_1(a) = a^k$ and $\sigma_2(b) = b^s$. One can easily verify that

$$a*(b*c) = a^k b^{ks} c^{s^2}$$

$$(a*b)*c = a^{k^2} b^{ks} c^s$$

which shows that the associative law of multiplication is no longer true. Denote the groupoid obtained by this distortion by the symbol $GQ_{p^n}(k, s)$. Though $GQ_{p^n}(k, s)$ is in general a groupoid, still it becomes quasi-group in special cases. In order that the equation $a*x = b$ may be solvable in it, it is necessary and sufficient that the equation $x^s = a^{-k}b = c$ must be solvable in the Galois field. It is well known from the theory of Galois fields that every element c is an sth. power if s has no common factor with $p^n - 1$. In other cases the equation $x^s = c$ has a solution only for some c. Hence if s is relatively prime to $p^n - 1$, the equation $a*x = b$ is solvable and similarly if k is relatively prime to $p^n - 1$, the equation $y*a = b$ is solvable. Thus $GQ_{p^n}(k, s)$ is a quasi-group if both k and s are relatively prime to $p^n - 1$.

Consider now the identity elements in the quasi-group $GQ_{p^n}(k, s)$. Let 1 denote the identity element of the Galois field. Then

$$1*b = 1^k b^s = b^s \text{ as } 1^k = 1.$$

Thus unless $s = 1$, 1 is no longer the common identity element. But if $s = 1$, then 1 will be the common left identity of the quasi-group. $GQ_{p^n}(k, 1)$ is therefore a left loop. In a similar manner it can be proved that $GQ_{p^n}(1, s)$ is a right loop. Combining the two results, one gets that $GQ_{p^n}(1, 1)$ is a loop; but if $k = 1$ and $s = 1$, $a*b = ab$ and hence $GQ_{p^n}(1, 1)$ becomes identical with the Galois field GF_{p^n}.

One important remark that will be useful later will be made here. In above, only the multiplication has been distorted. If the addition is kept undistorted, then the system $GQ_{p^n}(k, s)$ is a system of double composition which is additively a group and multiplicatively a quasigroup. It is important therefore to find whether the distributive law remain true. Now it is well known that in a Galois field of characteristic p the usual binomial theorem holds; but in the particular case when the power is p^m, $(a+b)^{p^m} = a^{p^m} + b^{p^m}$. Hence if $s = p^{m_s}$,

$$a*(b+c) = a^k(b+c)^s = a^k(b^s + c^s) = a^k b^s + a^k c^s = a*b + a*c$$

Thus only in $GQ_{p^n}(k, p^m)$ the left distributive law holds. Similarly only in $GQ_{p^n}(p^m, s)$ the right distributive law holds. Hence the both distributive laws hold in the quasi-group $GQ_{p^n}(p^m, p^m)$.

If the addition is also distorted, then the system $GQ_{p^n}(k, s)$ will be quasi-groups both additively and multiplicatively. Such a system will be called a QQ-system Take, for example, a new addition $a \dot{+} b = a-b$. Then

$$a^*(b \dot{+} c) = a^k(b \dot{+} c)^s = a^k(b-c)^s,$$

and hence conclusions about the distributive law similar to above can be deduced. Again apply another distortion $a \dot{+} b = ra+tb$, where r and t are two given integers. Then the distorted system will be denoted by the symbol $GQ_{p^n}(k, s, r, t)$. In general, it is additively a groupoid; but if $r \neq hp$ and $t \neq gp$, p being the characteristic, it is a quasi-group. In this case it becomes a QQ-system. Here $a^*0 = 0^*a = 0$. Further, 0 is no longer the common additive identity element in general. Indeed,

$$0 \dot{+} a = r0+ta = ta \neq a$$

$$a \dot{+} 0 = ra+t0 = ra \neq a$$

unless $r = hp+1$, $t = gp+1$, h and g being arbitrary integers. But in this case $a \dot{+} b = a+b$. The distributive law holds no longer unless $r = 1$, $t = 1$.

It has been obtained above that $GQ_{p^n}(k, s)$ is not a loop unless it is identical with the Galois field itself. So, it will be of some interest to construct from the Galois field GF_{p^n} quasi-groups which are loops distinct from the Galois field itself.

For this purpose, apply the distortion (defining the new product):

$$\begin{aligned} a^*b &= a^k b^{s_2} \text{ if } a \neq r^2 \\ &= a^k b^{s_1} \text{ if } a = r^2. \end{aligned}$$

Denoting the quasigroup obtained by the distortion as $GQ_{p^n}(k, s_1, s_2)$ it can be easily proved in the same way as before that

(1) $GQ(k, 1, s_2)$ are left loops

(2) $GQ(1, s_1, s_2)$ are right loops

(3) $GQ(1, 1, s_2)$ are loops.

The last mentioned loops are distinct from the Galois field GF_{p^n}. These cases were first discussed by L. E. Dickson (1905).

3. Systems of Double Composition

In the previous article certain systems have been obtained in which both addition and multiplication are defined. It will therefore be interesting to consider the different systems that may possibly have two compositions, namely the addition and the multiplication. It is well known that a system is a group if the following conditions are satisfied:

(1) ab belongs to the system,

(2) $a(bc) = (ab)c$, *i.e.*, the associative law holds.

(3) $ax = b$, $ya = b$ are uniquely solvable.

It is, of course, enough to take a less restricted form of (3), the equations $ax = b$ and $ya = b$ have at least one solution each. Now, it is possible, by taking some or all of these conditions, to get four kinds of systems:

(*i*) Systems in which only (1) holds.
These are called groupoids.

(*ii*) Systems in which (1) and (2) hold.
These are called semigroups.

(*iii*) Systems in which (1) and (3) hold.
These are called quasigroups.

(*iv*) Systems in which all of the conditions hold.
These are, as already stated, groups.

The composition can be taken either as addition or as multiplication even though it has been taken as multiplication in the above definition. Accordingly, the groupoids are additive or multiplicative.

If a system has two compositions, one will be taken to be the addition and the other will be taken to be the multiplication. Of course which one is the addition and which one is the multiplication is quite arbitrary. There are two options; one is called the dual of the other. Thus if the addition is replaced by the multiplication and vice versa, then a system is changed to its dual system. The correspondence

$$a+b \leftrightarrow ab$$

is the duality transformation.

A convention will now be made. A system which is a groupoid additively and a semigroup multiplicatively will be denoted by the symbol G_dS where G_d signifies a groupoid and S signifies a semigroup. Similarly it will be supposed that groups and quasigroups will be represented by the letters G and Q respectively. Then the systems of double compositions will be represented by a pair of letters like GS, GQ, GG_d etc. The first letter will state the nature of the system with respect to the addition and the second with respect to the multiplication.

Sixteen types of systems are possible:—

$$\begin{array}{cccc}
G_dG_d & G_dS & G_dQ & G_dG \\
SG_d & SS & SQ & SG \\
QG_d & QS & QQ & QG \\
GG_d & GS & GQ & GG
\end{array}$$

The four systems of the main diagonal line of this scheme are self dual. Systems which lie symmetrically about this diagonal are dual to each other.

Thus GS is dual of SG. The two operations in a system, say a GS system, may or may not be connected by a distributive law or by any such law. For example a GS-system with both sided distributive law is a ring. A lattice is an SS-system; a field is a GG-system.

Systems in which the associative law is not satisfied are systems like G_dQ, SQ, QQ, GQ and their duals. Here these non-associative systems will be especially considered. They have a great many interesting properties.

Non-associative systems have been studied very little, though some examples were known even at the time of Cayley, who came across non-associative systems in his study of biquaternions. Later Dickson proved that this system is a quasigroup multiplicatively. Let $q_1e_1+q_2e_2 = Q$ be a biquaternion where q_1 and q_2 are ordinary quaternions and $e_2^2 = -e_1$. Then it can be easily proved that $Q(RS) \neq (QR)S$ and that the equations $QX = R$ and $YQ = R$ are solvable. More recently P. Jordan has shown that a real GG_d-system is of great importance in Quantum Mechanics. Thus the subject has assumed a greater importance now-a-days.

4. INCLUSION RELATION.

In this article, the identity elements of a commutative quasi-groups will be dealt with. As the quasi-group Q is commutative, the left identity is identical with the right identity of each element of Q. These both sided identity relations are best studied by considering them as inclusion relations.

If a is the identity element of b, *i.e.*, $ab = b$, then this relation will be denoted as $a \geq b$ (to be read: a includes b). As the quasigroup Q is commutative, the two relations $ab = b$ and $ba = b$ implies that $a = b$. In terms of the inclusion relation this can be stated as a law of symmetry of this inclusion: if $a \geq b$ and $b \geq a$, then $a = b$. But this inclusion is not transitive. For if it was transitive, then $a \geq b$ and $b \geq c$ would imply $a \geq c$. Thus $ab = b$, $bc = c$ and $ac = c$; so that $ac = a(bc)$ and $bc = (ab)c$. But as $c = ac = bc$,

$$a(bc) = (ab)c.$$

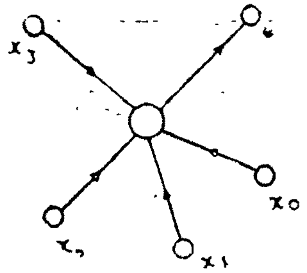
This can be true for any three elements a, b, c only if the associative law holds in the quasigroup. But then Q will reduce to a group and there will exist a common identity in it.

The inclusion relation $a \geq b$ can also be represented in the following manner:—

Let a circle represent the element a and another circle represent b. Then join the two circles by a directed segment, the direction being from the container a to the contained b. The whole figure thus obtained is the graph of the inclusion relation $a \geq b$. Graphs of all inclusion relations defined in a quasigroup taken together will be called the graph of the quasigroup. As the inclusion relation is not transitive, every container and contained must be joined by a segment and an indirect connection through a third element is not permissible. Further it follows from the symmetry property of the inclusion that if a is joined to b and b is joined to a, then $a = b$, *i.e.*, this cycle of two segments should be non-existent. a is then an idempotent element.

This inclusion relation has interesting properties. Given any element a, there exists only one element e such that $ea = a$; but there may exist no, one or many elements x such that ~~$xa = a$~~ $ax = x$. Thus a is included in only one element e, but a may include more than one element~~s~~. In the graph of these inclusion relations, consider the circle representing a as a tank to be filled with water and then this tank has many inlets but only one outlet.

The number of these inlets, *i.e.*, the number of solutions of the equation $xa = a$ will be called the weight of a and will be denoted by the symbol $w(a)$. ?



If $w(a) = 0$, then a has no inlets; but it must have a outlet. a can therefore be considered as a source element. The elements of this type will be important for our purpose.

If $w(a) = 1$, then a has only one inlet and only one outlet and so no water can be accumulated there. a can be called as a non-storage element. If, however, a is idempotent, then no water flows in or out of it and hence a is a stagnant element. Thus if every element of a quasigroup is idempotent, then the graph of this quasigroup will consist of circles having one link between any two of them. For example, the graph of the quasigroup whose multiplication table is

	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

consists of three circles with no link between them.

If $w(a) = r$, where r is any integer greater than 1 and may also be infinite, then water will be stored into it and the element a is a storage.

If a quasigroup is a loop, every element have the same element e as the identity element. Hence the graph of the loop will be a figure in which e lie in the centre and other elements around it. Water from all other tanks flows into e.

A given quasigroup determines its graph uniquely; but is the converse true? Does a given graph determine a quasigroup uniquely? The answer is evidently in the negative. Two groups of the same order (finite) will have the same graph, yet they will be distinct. The next question that comes in is: do the graphs determine classes of quasigroups? No attempt will be made to find the answer now. Only one remark will be made here. Consider, for example, the quasigroup

	a	b	c
a	b	a	c
b	a	c	b
c	c	b	a

Here $$ab = a,\ bc = b,\ ca = c,\ i.e.,\ a \leq b \leq c.$$

The graph is therefore an orientated triangle The graph determines the quasigroup uniquely. For if the inclusion relations are given, the elements in the positions (12), (23), (31) of the multiplication table are determined. As the quasigroup should be commutative, the positions (21), (32), (13) are filled with respective elements. The diagonal positions are still vacant. As every row and column of a quasigroup must contain all the elements, the elements in the diagonal positions are uniquely determined. The multiplication table is therefore fully known.

Let us now extend this notion of inclusion to a QQ-system. If in this system b is the additive identity of a, then this relation will be denoted by the inclusion relation

$$a \sqsubseteq b.$$

In general, this inclusion is distinct from the multiplicative inclusion $a \leq b$. But in perticular, one may be the converse of the other.

If the additive inclusion is the converse of the multiplicative inclusion, then $a \geq b$ implies $b \sqsupseteq a$ and hence $ab = b$ gives $a+b = a$. Thus

$$a+ab = a, \quad (a+b)b = b.$$

The necessary and the sufficient conditions that the additive inclusion in a QQ-system is the converse of the multiplicative inclusion is that the equations

$$a+ax = a, \quad ax = x$$

as well as the dual pair of them

$$a(a+x) = a, \quad a+x = x$$

must have a common solution for every a of the system.

For, if $a \geq b$ and $a \sqsubseteq b$ it has been proved above that

$$a+ab = a, \quad ab = b$$

i.e., the first pair has a solution. Conversely, let the pair of equations has a common solution b; then

$$a+ab = a, \quad ab = b$$

i.e., $ab = b$ and $a+b = a$ which proves that $a \geq b$ and $a \sqsubseteq b$. Dually, the equations

$$a(a+x) = a, \quad a+x = x$$

have a common solution.

The conditions require that the additive weight $w_1(a)$ and the multiplicative weight $w_2(a)$ must be greater than one. If $w_2(a) = 0$, then a is a source element of the multiplicative inclusion and must therefore be a sink element of the additive inclusion. Otherwise, there would exist an element b additively containing a but not being contained multiplicatively in a. This is impossible as every additive inclusion implies a multiplicative inclusion.

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ON A GENERALISATION OF LEGENDRE POLYNOMIALS

By

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1. Kharadze (1935) has recently generalised the Legendre polynomials by replacing the integral by the functional operator

$$I_1^{(k)} f(z)dz = F(1) + \alpha^{k-1}F(\alpha) + \cdots + \alpha F(\alpha^{k-1}) \tag{1.1}$$

where $F'(z) = f(z)$ and $1, \alpha, \alpha^2, \ldots, \alpha^{k-1}$ are the roots of $z^k - 1 = 0$; k being a positive integer. For $k = 2$,

$$I_1^{(k)} f(z)dz = \int_{-1}^{1} f(z)dz.$$

Kharadze (1942) has given the corresponding Rodrigue's formula and the differential equations corresponding to the polynomials $Q_n(z)$, which satisfy the relation

$$I_1^{(k)} Q_n(z)\varphi(z)dz = 0 \tag{1.2}$$

where $\varphi(z)$ is an arbitrary polynomial of degree $n-1$.

The object of this paper is to prove the results of Kharadze and to obtain some interesting relations between the polynomials $Q_n(z)$, in analogy with Legendre polynomials. On taking the differential equation as starting point, we obtain the second solution, and its relation with $Q_n(z)$. Also the associated functions corresponding to associated Legendre functions are obtained and their relations with Bessel functions are deduced. Lastly, we further generalise the polynomials $Q_n(z)$ in analogy with the ultraspherical polynomials (Szēgo, 1939).

2. From (1.1) it follows that

$$I_1^{(k)} z^{mk} dz = k/(km+1) \quad \text{and} \quad I_1^{(k)} z^r dz = 0,$$

when r is an integer $\neq mk$ for any value of m. From this we derive by an elementary method the following

For $n = km$, $Q_n(z)$ has only terms of the type z^{kr} and for $n = km+1$, $Q_n(z)$ has only terms of the type z^{kr+1}; for other values on n, $Q_n(z)$ vanishes identically.

The following theorems are easy to establish.

Theorem I. *The following linear relation holds between* $Q_{km}(z)$, $Q_{km-k+1}(z)$ *and* $Q_{km-k}(z)$ for $k \geq 2$.

$$Q_{km}(z) = Az^{k-1}Q_{km-k+1}(z) - C.Q_{km-k}(z)$$

where

$$A = \frac{a_{km}}{a_{km-k+1}} \quad \text{and} \quad C = \frac{a_{km}a_{km-k+1}^{(2)} - a_{km}^{(2)}a_{km-k+1}}{a_{km-k}\,a_{km-k+1}} > 0.$$

We denote by a_{km} the coefficient of the term of the highest power in $Q_{km}(z)$ and $a_{km}^{(2)}$ the coefficient of the next highest term in $Q_{km}(z)$. Similarly for a_{km-k+1}, $a_{km-k+1}^{(2)}$ etc.

Theorem II. *The following linear relation holds between* $Q_{km+1}(z)$, $Q_{km}(z)$ *and* $Q_{km-k+1}(z)$:

$$Q_{km+1}(z) = A_1 z\, Q_{km}(z) - C_1\, Q_{km-k+1}(z)$$

where

$$A_1 = \frac{a_{km+1}}{a_{km}} \quad \textit{and} \quad C_1 = \frac{a_{km+1}\, a_{km}^{(2)} - a_{km}\, a_{km+1}^{(2)}}{a_{km}\, a_{km-k+1}} > 0.$$

If $Q_{km}(z)$ and $Q_{km+1}(z)$ satisfy the relations (A) and (B) *viz.*,

$$I_1^{(k)} \{Q_{km}(z)\}^2 dz = 1 \tag{A}$$

$$I_1^{(k)} z^{k-2} \{Q_{km+1}(z)\}^2 dz = 1, \tag{B}$$

we obtain from Theorems I and II, the following:

Theorem III. *If* $\{Q_{km}(z)\}$ *and* $\{Q_{km+1}(z)\}$ *satisfy* (A) *and* (B), *then for any two points* z_0, z, *we have the following relation.*

$$\frac{a_{km}}{a_{km+1}} \cdot \frac{Q_{km+1}(z)Q_{km}(z_0) - Q_{km+1}(z_0)Q_{km}(z)}{z - z_0} = \sum_{r=0}^{m} Q_{rk}(z)\, Q_{rk}(z_0) + \frac{z^{k-1} - z_0^{k-1}}{z - z_0} \sum_{r=0}^{m-1} Q_{rk+1}(z)\, Q_{rk+1}(z_0)$$

(z^k and z_0^k being real).

Lastly we observe that the zeros of $Q_n(z)$ are n in number, and are symmetrically distributed on the lines joining the origin to the points, $1, \alpha, \alpha^2, \ldots, \alpha^{k-1}$.

3. We derive now the analogue of Rodrigues' formula for the polynomials $Q_n(z)$. We use the notation:

$$\int_1^z Q_{km}(z)dz = Q_{km}^{(-1)}(z), \quad \int_1^z z^{k-2} \int_1^z Q_{km}(z).(dz)^2 = Q_{km}^{(-k)}(z) \text{ and } \int_1^z Q_{km}^{(-rk)}(z)dz = Q_{km}^{(-rk-1)}(z).$$

Since

$$\int_0^1 Q_{km}(z).F(z)dz = 0$$

where $F(z)$ is an arbitrary polynomial of degree $\leq km-k$ and contains terms of the type z^{kr}, we have on integrating by parts

$$Q_{km}^{(-1)}(0).F(0) + \int_0^1 Q_{km}^{(-1)}(z).F'(z)dz = 0.$$

Since $F(z)$ is arbitrary, we may take $F(z)$ to be a non-zero constant, and this gives $Q_{km}^{(-1)}(0) = 0$. We then have

$$\int_0^1 z^{k-2}.Q_{km}^{(-1)}(z).\frac{F'(z)}{z^{k-2}}dz = 0.$$

Again integrating by parts, we have

$$\left[\frac{F'(z)}{z^{k-2}} \int_1^z z Q_{km}^{(-1)}(z)dz\right]_{z=0} + \int_0^1 \left\{D_{(k)}^{(k)} F(z)\right\}\left(\int_1^z z^{k-2} \int_1^z Q_{km}(z).(dz)^2\right)dz = 0$$

where

$$D_{(k)}^{(k)} = \frac{d}{dz}\frac{1}{z^{k-2}}\frac{d}{dz}$$

and $D_{(k)}^{(rk)}$ means that the operator $D_{(k)}^{(k)}$ has been repeated r times. Also $D_{(k)}^{(rk+1)} \equiv \frac{d}{dz}D_{(k)}^{(rk)}$. Then on repeating the argument again and again, we have

$$Q_{km}^{(-rk-1)}(0) = 0, \quad r = 0, 1, \ldots, m.$$

Also

$$Q_{km}^{(-rk-1)}(1) = 0, \quad r = 0, 1, \ldots, m.$$

Then since $Q_{km}^{(-km-1)}(z)$ is a polynomial of degree $2km+1$, which vanishes together with

$$D_{(k)}^{(sk)}Q_{km}^{(-km-1)}(z) \quad (s = 0, 1, \ldots, m-1)$$

at $z = 0$ and 1, we have

$$Q_{km}(z) = (1/A_{km})D_{(k)}^{(km)}(z^k-1)^{2m} \tag{3.1}$$

Similarly

$$Q_{km+1}(z) = (1/A_{km+1})D_{(k)}^{(km+k-1)}(z^k-1)^{2m+1}. \tag{3.2}$$

If we choose $A_{km} = k^{2m}(2m)!$ and $A_{km+1} = k^{2m+1}(2m+1)!$ we get several interesting relations analogous to those for Legendre Polynomials.

In order to find the differential equation which $Q_{km}(z)$ satisfies, we put

$$u = (z^k-1)^{2m}.$$

Differentiate it and multiply the result by $(z^k-1)/z^{k-2}$; then we have

$$(u'/z^{k-2})(z^k-1) = 2mk.uz$$

Differentiating it again, we have

$$(z^k-1).D_{(k)}^{(k)}u + kz^{k-1}.(u'/z^{k-2}) = 2mk(u+u'z)$$

Applying the operator $D_{(k)}^{(mk)}$ to both sides, and putting $D_{(k)}^{(mk)}u = y$, we have

$$(z^k-1)D_{(k)}^{(k)}y + kzy' = mk(mk+1)y. \tag{3.3}$$

Similarly the differential equation satisfied by $Q_{km+1}(z)$ is

$$(z^k-1)y'' + kz^{k-1}y' = (mk+1)(mk+k)yz^{k-2}. \tag{3.4}$$

4. *Recurrence relations for $Q_{km}(z)$ and $Q_{km+1}(z)$:*

On using the Rodrigue's form for $Q_{km}(z)$ and $Q_{km+1}(z)$, we have

$$Q_{km}(z) - Q_{km-k}(z) = (2mk-k+1)\int_0^z z^{k-2}\,Q_{km-k+1}(z) \tag{4.1}$$

and

$$Q_{km+1}(z) - Q_{km-k+1}(z) = (2km+1)\int_0^z Q_{km}(z)dz. \tag{4.2}$$

On obtaining the coefficients of z^{km}, z^{km-k} in $Q_{km}(z)$ and the coefficients of z^{km+1} and z^{km-k+1} in $Q_{km+1}(z)$ by means of Rodrigue's formula, we have, from theorems I and II, the following relations:

$$kmQ_{km}(z) = (2mk-k+1)Q_{km-k+1}(z).z^{k-1}-(mk-k+1)Q_{km-k}(z) \tag{4 3}$$

and

$$(km+1)Q_{km+1}(z) = (2mk+1)z.Q_{km}(z)-kmQ_{km-k+1}(z) \tag{4.4}$$

From these we easily get the following recurrence relations:

$$\left.\begin{aligned} Q'_{km}(z) &= z^{k-1}Q'_{km-k+1}(z)+mkz^{k-2}Q_{mk-k+1}(z) \\ Q'_{km+1}(z) &= zQ'_{km}(z)+(km+1)Q_{km}(z) \\ zQ'_{km}(z)-Q'_{km-k+1}(z) &= kmQ_{km}(z) \\ z^{k-1}Q'_{km-k+1}(z)-Q'_{km-k}(z) &= (mk-k+1)z^{k-2}Q_{km-k+1}(z) \\ (z^k-1)Q'_{km}(z) &= kmz^{k-1}Q_{km}(z)-kmz^{k-2}Q_{km-k+1}(z) \\ \text{and}\quad (z^k-1).Q'_{km+1}(z) &= (mk+1)z^{k-1}Q_{km+1}(z)-(mk+1)Q_{km}(z) \end{aligned}\right\} \tag{4.4'}$$

Also

$$I_1^{(k)}\{Q_{km}(z)\}^2dz = k\int_0^1\{Q_{km}(z)\}^3dz = \frac{ka_{km}}{k^{2m}(2m)!}\int_0^1 z^{km}\{D_{(k)}^{(km)}(z^k-1)^{3m}\}dz,$$

where a_{km} is the coefficient of z^{km} in $Q_{km}(z)$.

Since

$$a_{km} = \frac{(2mk-k+1)\,(2mk-2k+1)\;.\;.\;.\;(mk+1)}{k^m.m!}$$

we have

$$I_1^{(k)}\{Q_{km}(z)\}^2dz = \frac{k}{2km+1}. \tag{4.5}$$

Similarly

$$I_1^{(k)}z^{k-2}\{Q_{km+1}(z)\}^2dz = \frac{k}{2mk+k+1}. \tag{4,6}$$

Since $Q_{km}(z)$ satisfies the relation,

$$\int_0^1 Q_{km}(z)\,z^{rk}dz = 0,\quad r = 0, 1, \ldots m-1$$

then from (4.5) we obtain on putting $z^k = u$, and $Q_{km}(z) = S_m(z^k)$ the following properties of $S_m(u)$:

$$\int_0^1 S_m(u).u^r.u^{1/k-1}.du = 0,\quad r = 0, 1, \ldots m-1$$

and

$$\int_0^1 S_m^2(u).u^{1/k-1}du = \frac{k}{2mk+1}.$$

Changing the interval of integration to $(-1, 1)$, by putting $u = \frac{1}{2}(1+x)$, and on comparing with the Jacobi's polynomial we have

$$Q_{km}(z) = P_m^{(0,\,1/k-1)}(2z^k-1) = {}_2F_1(-m,\ m+1/k,\ 1;\ 1-z^k). \tag{4.7}$$

Similarly

$$Q_{km+1}(z) = z.P_m^{(0,\,1/k)}(2z^k-1) = z.{}_2F_1(-m,\ m+1/k+1,\ 1;\ 1-z^k). \tag{4.8}$$

Since in (4.7) if we replace m by $-m-1/k$, the hypergeometric form remains unchanged, we have

$$Q_{-km-1}(z) = Q_{km}(z).$$

Similarly

$$Q_{-km-k}(z) = Q_{km+1}(z).$$

5. On finding the solutions of (3.3) and (3.4) in series form we have

$$R_{km}(z) = A_1 z^{-mk-1}.{}_2F_1(m+1,\ m+1/k;\ 2m+1+1/k;\ z^{-k})$$

and

$$R_{km+1}(z) = Az^{-mk-k}.{}_2F_1(m+1,\ m+1+1/k;\ 2m+2+1/k;\ z^{-k})$$

where $R_{km}(z)$ and $R_{km+1}(z)$ are the second solutions of (3.3) and (3.4) respectively. Constants A_1 and A are to be suitably determined.

If we put

$$A_1 = \frac{k^m m!}{(2mk+1)\cdots(mk+k+1)(mk+1)}$$

and

$$A = \frac{k^m m!}{(2mk+k+1)(2mk+1)\cdots(mk+k+1)}$$

we have

$$R_{km}(z) = \int_0^1 \frac{z^{k-1}Q_{km}(t)}{z^k-t^k}dt \tag{5.1}$$

and

$$R_{km+1}(z) = \int_0^1 t^{k-1}\,\frac{Q_{km+1}(t)}{z^k-t^k}dt. \tag{5.2}$$

To prove (5.1) we observe that

$$\int_0^1 Q_{km}(t).t^{mk+rk}\,dt = \frac{1}{k^{2m}(2m)!}\int_0^1 t^{mk+rk}.\{D_{(k)}^{(mk)}\,(t^k-1)^{2m}\}.dt$$

which is equal to

$$\frac{1}{k}.\frac{\Gamma(m+r+1)\Gamma(m+r+1/k)}{\Gamma(r)\Gamma(2m+r+1/k+1)}.$$

On expanding $1/(z^k-t^k)$ in an infinite series and on integrating term by term, which is justified if $|z|>1$, we get (5.1). Similarly (5.2) is obtained.

From (5.1) and (5.2) we see that $R_{km}(z)$ and $R_{km+1}(z)$ satisfy all the recurrence relations satisfied by $Q_{km}(z)$ and $Q_{km+1}(z)$.

From the differential equations we have also

$$R_{km}(z) = Q_{km}(z)\int_z^\infty \frac{z^{k-2}}{(z^k-1)Q_{km}^2(z)}dz$$

and

$$R_{km+1}(z) = Q_{km+1}(z)\int_z^\infty \frac{1}{(z^k-1)Q^2_{km+1}(z)}dz.$$

6. Since $z^{k-1}/(z^k-t^k)$ contains terms in powers of t^k, we may express it in a series of $Q_{km}(t)$ under certain conditions. For if

$$\frac{z^{k-1}}{z^k-t^k} = \sum_{m=0}^{\infty} A_{km}(z).Q_{km}(t) \tag{6.1}$$

where

$$A_{km}(z) = (2km+1)R_{km}(z)$$

we have by means of the recurrence-relations for $R_{km}(z)$ and $Q_{km}(z)$:

$$\sum_{r=0}^{\infty}(2rk+1)(z^k-t^k)Q_{rk}(t)R_{rk}(z) = \frac{(km+1)(km+k)}{2mk+k+1}[R_{km+k}(z)Q_{km}(t)-Q_{km+k}(t)R_{km}(z)]$$

$$+\frac{k}{k+1}[R_0(z)Q_k(t)-Q_0(t)R_k(z)]+(z^k-t^k)R_0(z)Q_0(t).$$

Since

$$kQ_k(t) = (k+1)Q_1(t)t^{k-1}-Q_0(t)$$

and

$$kR_k(z) = (k+1)R_1(z)z^{k-1}-R_0(z)$$

we find by means of the hypergeometric expression of $R_0(z)$ that

$$\frac{k}{k+1}[R_0(z)Q_k(t)-Q_0(t)R_k(z)]+(z^k-t^k)R_0(z)Q_0(t) = z^{k-1}.$$

Hence

$$\left|z^{k-1}-\sum_{r=0}^{m}(2rk+1)(z^k-t^k)Q_{rk}(t)R_{rk}(z)\right| = \frac{(km+1)(km+k)}{2mk+k+1}\,|R_{km+k}(z)Q_{km}(t)-Q_{km+k}(t)R_{km}(z)| \tag{6.2}$$

Since

$$Q_{km}(t) = O(m^{\frac{1}{2}})$$

and

$$|R_{km}(z)| \leq \int_0^1 t^m(1-t)^{m+1/k-1}|(z^k-t)^{-m-1/k}|\,dt$$

$$\leq \rho^{-m-1/k}\frac{\Gamma(m+1)\Gamma(m+1/k)}{\Gamma(2m+1/k+1)}$$

where $\rho = |z|^k-1$, we find that the right hand side in (6.2) tends uniformly to zero, provided $\rho > \frac{1}{4}$ *i.e.*, $|z|^k > \frac{5}{4}$.

Under this condition the expression (6.1) holds good, Similarly,

$$\frac{t^{k-1}}{z^k-t^k} = \sum_{m=0}^{\infty}(2mk+k+1)Q_{km+1}(t)R_{km+1}(z).$$

7. We observe that in the following integral,

$$I = \frac{1}{k}\int_0^1 t^m(1-t)^{m+1/k-1}(z^k-t)^{-m-1/k}\,dt$$

if $|z|>1$, we may expand $(1-t/z^k)^{-m-1/k}$ and integrate term by term.* This gives on simplification

$$I=\frac{1}{k\,z^{km+1}}\,\frac{\Gamma(m+1)\Gamma(m+1/k)}{\Gamma(2m+1+1/k)}\;{}_2F_1(m+1,\ m+1/k,\ 2m+1+1/k;\ 1/z^k)=R_{km}(z).$$

Similarly, for $R_{km+1}(z)$ we have the following integral representation:

$$R_{km+1}(z)=\frac{1}{k}\int_0^1 t^{m+1/k}(1-t)^m(z^k-t)^{-m-1}dt, \tag{7.1}$$

provided $|z|>1$.

8. From the differential equation (3.3) on applying the operator $D_{(k)}^{(k)}$ n times to both sides, we have

$$(1-z^k)\,D_{(k)}^{(n+1)k}y-(2n+1)kz.D_{(k)}^{(nk+1)}y+[mk(mk+1)-n^2k(k-1)-n(n+1)k]D_{(k)}^{(nk)}y=0.$$

On putting $D_{(k)}^{(nk)}y=v$, we have

$$(1-z^k)D_{(k)}^{(k)}v-(2n+1)kzv'+[mk(mk+1)-n^2k(k-1)-n(n+1)k]v=0 \tag{8.1}$$

Putting $v=(1-z^k)^{-n}w$, we have after some simple calculations the following equation in w:

$$(1-z^k)D_{(k)}^{(k)}w-kzw'+w\left[mk(mk+1)-\frac{n^2k^2}{1-z^k}\right]=0 \tag{8.2}$$

The complete solution of this is given by

$$w=a(1-z^k)^n.D_{(k)}^{(nk)}Q_{km}(z)+b(1-z^k)^nD_{(k)}^{(nk)}R_{km}(z).$$

If we differentiate (8.1) and divide the result by z^{k-2}, we have

$$\begin{aligned}(1-z^k)D_{(k)}^{(n+2)k-1}\,y-(2n+2)kz.D_{(k)}^{(n+1)k}\,y&\\ +[mk(mk+1)-(n+1)^2k(k-1)-n(n+1)k]D_{(k)}^{(n+1)k-1}\,y&=0.\end{aligned}$$

On putting

$$D_{(k)}^{(n+1)k-1}\,y=(1-z^k)^{-(nk+1)/k}.w$$

we have after some simplification

$$(1-z^k)w''-2(k-1)z^{k-1}w'+wz^{k-2}\left[(mk+k-1)(mk-k+2)-\frac{(nk+1)\,(nk+k-1)}{1-z^k}\right]=0 \tag{8.3}$$

A complete solution of this is

$$w=a(1-z^k)^{(nk+1)/k}\,D_{(k)}^{(nk+k-1)}\,Q_{km}(z)+b(1-z^k)^{(nk+1)/k}\,D_{(k)}^{(nk+k-1)}\,R_{km}(z).$$

On considering the series solutions of the above differential equations we have

$$Q_{km}^{kn}(z)=\frac{(-1)^{m-n}\Gamma(m+n+1/k)}{\Gamma(m-n+1)\Gamma(1/k)}k^{2n}(1-z^k)^n.{}_2F_1(n-m,\ m+n+1/k;\ 1/k;\ z^k) \tag{8.4}$$

and

$$\begin{aligned}Q_{km}^{kn+1}(z)=\frac{(-1)^{m-n-1}\Gamma(m+n+1/k+1)}{\Gamma(m-n)\Gamma(1/k)}\,k^{2n+2}\,z(1-z^k)^{(nk+1)/k}&\\ \times{}_2F_1(n+1-m,\ m+n+1+1/k;\ 1/k+1;\ z^k)&\end{aligned} \tag{8.5}$$

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* Term by term integration is easily justifiable.

Also*

$$R_{km}^{kn}(z) = \frac{-k^{m+n}(m+n)\,!}{(mk+1)\ .\ .\ .\ (2mk+1)}\{(mk+1)(mk+k+1)\ .\ .\ .\ (mk+nk-k+1)\}$$

$$\times(1-z^k)^n z^{-mk-nk-1}\,.\,{}_2F_1(m+n+1,\ m+n+1/k\,;\ 2m+1/k+1;\ 1/z^k)$$

and

$$R_{km}^{kn+1}(z) = \frac{k^{m+n}(m+n)\,!}{(mk+1)\ .\ .\ .\ 2mk+1)}\{(mk+1)\ .\ .\ .\ (mk+nk+1)\}$$

$$\times(1-z^k)^{(nk+1)/k} z^{-mk-nk-k}\,.\,{}_2F_1(m+n+1,\ m+n+1+1/k\,;\ 2m+1+1/k\,;\ 1/z^k)$$

Here $Q_{km}^{kn}(z)$ and $R_{km}^{kn}(z)$ are the first and second solutions of (8.2). Similarly $Q_{km}^{(kn+1)}(z)$ and $R_{km}^{kn+1}(z)$ correspond to (8.3) The constant factors in (8.4) and (8.5) are obtained on comparing the terms independent of z; this is made easy by means of the following expressions analogous to Rodrigue's formula:

$$Q_{km}^{kn}(z) = \frac{1}{k^{2m}(2m)\,!}(1-z^k)^n D_{(k)}^{(km+kn)}(1-z^k)^{2m}$$

$$Q_{km}^{kn+1}(z) = \frac{1}{k^{2m}(2m)\,!}(1-z^k)^{(nk+1)/k} D_{(k)}^{(km+kn+k-1)}(1-z^k)^{2m}.$$

By means of the differential equation (8.1) we find that $Q_{km}^{kn}(z)$ and $Q_{kr}^{kn}(z)$ satisfy the relation

$$\int_0^1 Q_{km}^{kn}(z)\,.\,Q_{kr}^{kn}(z)\,dz = 0.$$

9. *An Identity.* We now prove the following identity:

$$(1-z^k)^{2n} D_{(k)}^{(km+kn)}(1-z^k)^{2m} \equiv \frac{k^{4n}\,.\,(m+n)\,!\ \Gamma(m+n+1/k)}{(m-n)\,!\ \Gamma(m-n+1/k)} \times D_{(k)}^{(km-kn)}(1-z^k)^{2m} \tag{9.1}$$

To prove this we observe that the coefficient of z^{kr} on the right is equal to

$$(-1)^{m-n+r}\frac{\Gamma(2m+1)\Gamma(m-n+r+1/k)k^{2m+2n}\Gamma(m+n+1)\Gamma(m+n+1/k)}{\Gamma(m+n-r+1)\Gamma(r+1)\Gamma(m-n+1)\Gamma(m-n+1/k)\Gamma(r+1/k)}. \tag{9.2}$$

Collecting the terms containing z^{kr} on the left, we have the coefficient of z^{kr} on the left

$$= k^{2m+2n}(-1)^{m+n+r}\frac{\Gamma(2m+1)\Gamma(2n+1)\Gamma(m+n+1/k)}{\Gamma(m-n+1)\Gamma(r+1)\,\Gamma(2n-r+1)\,\Gamma(1/k)}\,.\,{}_3F_2\left(\begin{matrix}-r, & m+n+1/k, & n-m\,; \\ & 2n-r+1, & 1/k\end{matrix}\right) \tag{9.3}$$

By a formula of Saalschütz (1809), we have

$${}_3F_2\left(\begin{matrix}-r, & m+n+1/k, & n-m\,; \\ & 2n-r+1, & 1/k\end{matrix}\right) = \frac{\Gamma(2n-r+1)\Gamma(1-1/k-r)\Gamma(m+n+1)\Gamma(1-1/k+n-m)}{\Gamma(1-1/k)\Gamma(2n+1)\Gamma(n-m-r+1-1/k)\Gamma(m+n-r+1)}.$$

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* For $n=0$, we take unity for $\{(mk+1)(mk+k+1)\ .\ .\ (mk+nk-k+1)\}$.

Using the known relation

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

we find on simplification that (9.3) reduces to (9.2). The identity is then established.

By means of the identity we have

$$\begin{aligned}\int_0^1 \{Q_{km}^{kn}(z)\}^2 dz &= \frac{1}{k^{4m}((2m)!)^2}\int_0^1 (1-z^k)^{4n}[D_{(k)}^{(km+kn)}(1-z^k)^{2m}]^2 dz \\ &= \frac{k^{4n-4m}}{\{(2m)!\}^2}\frac{(m+n)!\,\Gamma(m+n+1/k)}{(m-n)!\,\Gamma(m-n+1/k)}\int_0^1 \{D_{(k)}^{(km+kn)}(1-z^k)^{2m}\}\{D_{(k)}^{(km-kn)}(1-z^k)^{2m}\}dz \\ &= \frac{k^{4n}}{2km+1}\frac{(m+n)!\,\Gamma(m+n+1/k)}{(m-n)!\,\Gamma(m-n+1/k)}. \end{aligned} \tag{9.3'}$$

We also see that the following recurrence-relation holds good:

$$z^{k-2}Q_{km}^{kn+1}(z) = (1-z^k)^{1/k}\frac{d}{dz}Q_{km}^{kn}(z) + knz^{k-1}(1-z^k)^{-(k-1)/k}Q_{km}^{kn}(z). \tag{9.4}$$

The relation can be seen to hold true on using the hypergeometric expressions for $Q_{km}^{kn}(z)$ and $Q_{km}^{kn+1}(z)$.

10. *Relation between Bessel functions and* $Q_{km}^{kn}(z)$ and $Q_{km}^{kn+1}(z)$:

If in (8.3) we change the independent variable z by means of the substitution

$$x^k = m^2k^2(1-z^k) \tag{10.1}$$

then the transformed equation becomes on making $m \to \infty$,

$$\frac{d^2w}{dx^2} + \frac{1}{x}\frac{dw}{dx} + x^{k-2}\left(1 - \frac{n^2k^2}{x^k}\right)w = 0.$$

Again, making the substitution

$$x = (\tfrac{1}{2}ky)^{2/k}$$

the above equation reduces to

$$\frac{d^2w}{dy^2} + \frac{1}{y}\frac{dw}{dy} + \left(1 - \frac{4n^2}{y^2}\right)w = 0.$$

Hence, we have

$$\lim_{m\to\infty} Q_{km}^{kn}\left\{\left(1 - \frac{x^k}{m^2k^2}\right)^{1/k}\right\} = J_{2n}\left(\frac{2}{k}x^{k/2}\right) \tag{10.2}$$

Similarly if in (8.4) we make the substitution (10.1) we have as $m \to \infty$,

$$\frac{d^2w}{dx^2} + \frac{k-1}{x}\frac{dw}{dx} + wx^{k-2}\left[1 - \frac{(nk+1)(nk+k-1)}{x^k}\right] = 0.$$

Put

$$w = x^{-(k-2)/2}v,$$

Then we have

$$\frac{d^2v}{dx^2}+\frac{1}{x}\frac{dv}{dx}+x^{k-2}v.\left[1-\frac{(nk+k/2)^2}{x^k}\right]=0.$$

Then as before we have

$$\lim_{m\to\infty} Q_{km}^{kn+1}\left[\left(1-\frac{z^k}{m^2k^2}\right)^{1/k}\right]=x^{-(k-2)/2}J_{2n+1}\left(\frac{2}{k}x^{k/2}\right). \tag{10.3}$$

11. We consider now polynomials given by

$$(1-z^k)^\lambda Q_{km}^{(\lambda)}(z)=\frac{1}{A_{km}}D_{(k)}^{(km)}(1-z^k)^{2m+\lambda} \tag{11.0}$$

and

$$(1-z^k)^\lambda Q_{km+1}^{(\lambda)}(z)=\frac{1}{A_{km+1}}D_{(k)}^{(km+k-1)}(1-z^k)^{2m+1+\lambda}. \tag{11 0.0}$$

By the method of Art. 3, we find that $Q_{km}^{(\lambda)}$ satisfies the differential equation:

$$(1-z^k)\,D_{(k)}^{(k)}\,y-(\lambda+1)kzy'+y'mk(mk+\lambda k+1)y=0 \tag{11.1}$$

and for $Q_{km+1}^{(\lambda)}(z)$, we have the following:

$$(1-z^k)y''-kz^{k-1}(\lambda+1)y'+y(mk+1)(mk+\lambda k+k)z^{k-2}=0 \tag{11.2}$$

Also

$$\int_0^1 Q_n^{(\lambda)}(z)(1-z^k)^\lambda z^{kr}dz=0;\ \ (r=0,1,\ldots,n-1,\ \ n=km\ \text{ or }\ km+1)$$

follows from (11.0). This gives the following hypergeometric form for $Q_{km}^{(\lambda)}(z)$:

$$Q_{km}^{(\lambda)}(z)=\binom{2m+2\lambda}{2m}\cdot F(-m,\ m+\lambda+1/k,\ \lambda+1;\ 1-z^k).$$

Similarly,

$$Q_{km+1}^{(\lambda)}(z)=\binom{2m+2\lambda+1}{2m+1}z\ F(-m,\ m+\lambda+1/k+1,\ \lambda+1;\ 1-z^k).$$

From these, on comparison we have

$$\frac{1}{A_{km}}=\frac{1}{k^{2m}}\cdot\frac{\Gamma(2m+2\lambda+1)\Gamma(\lambda+1)}{\Gamma(2m+1)\Gamma(2m+\lambda+1)\Gamma(2\lambda+1)}$$

and

$$\frac{1}{A_{km+1}}=\frac{(-1)}{k^{2m+1}}\cdot\frac{\Gamma(2m+2\lambda+2)\Gamma(\lambda+1)}{\Gamma(2m+2)\Gamma(2\lambda+1)\Gamma(2m+\lambda+2)}.$$

The following recurrence-relations can be determined from Theorem II and III:

$$km(2m-1)Q_{km}^{(\lambda)}(z)=(2m-1)(2mk+\lambda k-k+1)z^{k-1}Q_{km-k+1}^{(\lambda)}(z)$$
$$-(mk-k+1)(2m+2\lambda-1)Q_{km-k}^{(\lambda)}(z) \tag{11.3}$$

and

$$(2m+1)(mk+\lambda k+1)Q_{km+1}^{(\lambda)}(z)=(2km+\lambda k+1)(2m+2\lambda+1)zQ_{km}^{(\lambda)}(z)$$
$$-(mk+\lambda k)(2m+2\lambda+1)Q_{km-k+1}^{(\lambda)}(z) \tag{11.4}$$

Also by means of the hypergeometric forms of $Q_{km}^{(\lambda)}(z)$ and $Q_{km+1}^{(\lambda)}(z)$, we have

$$\frac{d}{dz} Q_{km}^{(\lambda)}(z) = \frac{mk+\lambda k+1}{2m+2\lambda+1} z^{k-2}(2\lambda+1) Q_{km-k+1}^{(\lambda+1)}(z)$$

and

$$\frac{d}{dz} Q_{km+1}^{(\lambda)}(z) = \frac{km+1}{2m+1}(2\lambda+1) Q_{km}^{(\lambda+1)}(z).$$

Also we have

$$I_1^{(k)}\{Q_{km}^{(\lambda)}(z)\}^2(1-z^k)^\lambda\, dz = \left\{\frac{\Gamma(2m+2\lambda+1)\Gamma(\lambda+1)}{\Gamma(2m+1)\Gamma(2\lambda+1)}\right\}^2 \frac{\Gamma(m+1)\Gamma(m+1/k)}{\Gamma(m+\lambda+1)\Gamma(m+\lambda+1/k)} \cdot \frac{k}{2mk+\lambda k+1}$$

and

$$I_1^{(k)}\{Q_{km+1}^{(\lambda)}(z)\}^2(1-z^k)^\lambda z^{k-2}\, dz = \left\{\frac{\Gamma(2m+2\lambda+2)\Gamma(\lambda+1)}{\Gamma(2m+2)\Gamma(2\lambda+1)}\right\}^2$$

$$\times \frac{\Gamma(m+1)\Gamma(m+1/k+1)}{\Gamma(m+\lambda+1)\Gamma(m+\lambda+1/k+1)} \cdot \frac{k}{2mk+\lambda k+k+1}$$

The second solution of $R_{km}^{(\lambda)}(z)$ of the equation (11.1) is given by

$$R_{km}^{(\lambda)}(z) = \frac{A_1}{z^{mk+\lambda k+1}} \cdot {}_2F_1(m+\lambda+1,\ m+\lambda+1/k;\ 2m+\lambda+1/k+1;\ 1/z^k)$$

$$= \frac{(1-z^k)^{-\lambda}.A_2}{z^{km+1}} \cdot {}_2F_1(m+1,\ m+1/k;\ 2m+\lambda+1/k+1;\ 1/z^k).$$

The constants A_1 and A_2 being determined from the following:

$$(1-z^k)^\lambda R_{km}^{(\lambda)}(z) = \frac{1}{k} \cdot \frac{\Gamma(m+1)\Gamma(m+1/k+\lambda)}{\Gamma(2m+1+\lambda+1/k)} \int_0^1 \frac{z^{k-1} Q_{km}(t)}{z^k-t^k}(1-t^k)^\lambda dt.$$

We also see that the following integral representation holds true:

$$R_{km}^{(\lambda)}(z) = \frac{1}{k}(1-z^k)^{-\lambda} \int_0^1 t^m(1-t)^{m+1/k+\lambda-1}(z^k-t)^{-m-1/k}\, dt$$

$$= \frac{1}{k} \cdot \frac{(-1)^\lambda \Gamma(m+1)\Gamma(m+1/k+\lambda)}{\Gamma(m+\lambda+1)\Gamma(m+1/k)} \int_0^1 t^{m+\lambda}(1-t)^{m+1/k-1}(z^k-t)^{-m-\lambda-1/k} dt.$$

Similarly,

$$R_{km+1}^{(\lambda)}(z) = \frac{B_1}{z^{mk+\lambda k+k}} \cdot F(m+\lambda+1,\ m+\lambda+1/k+1,\ 2m+\lambda+1/k+2;\ 1/z^k)$$

$$= \frac{B_2(1-z^k)^{-\lambda}}{z^{mk+k}} F(m+1,\ m+1/k+1,\ 2m+\lambda+1/k+2;\ 1/z^k)$$

$$= \frac{\Gamma(2m+1)\Gamma(2\lambda+1)\Gamma(m+1+1/k+\lambda)}{\Gamma(2m+2+2\lambda)\Gamma(\lambda+1)\Gamma(m+1+1/k)} \int_0^1 \frac{t^{k-1}}{z^k-t^k} Q_{km+1}^{(\lambda)}(t).(1-t^k)^\lambda dt$$

$$= \frac{1}{k}(1-z^k)^{-\lambda} \int_0^1 t^{m+1/k}(1-t)^{m+\lambda}(z^k-t)^{-m-1}\, dt.$$

The values of B_1 and B_2 can be easily determined.

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CIRCLES OF DOUBLE CONTACT OF A BI-CIRCULAR QUARTIC

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INTRODUCTION

The present paper, as its name implies, deals with certain novel traits of a bicircular quartic with special reference to its aggregate of circles of double contact. As is well-known, bicircular quartics are, broadly speaking, classified as ***bicursal*** or *unicursal*, according as their deficiency (or genus) is 1 or 0. Unicursal bicircular quartics, derivable, as they are, from curves of the second degree by process of inversion, naturally admit of a comparatively simple treatment and will be accordingly ignored in the present paper. In point of fact, most of the properties of bicursal (bicircular quartics, when ***suitably*** modified, hold also for those of the unicursal type.

As a matter of convenience the paper has been divided into four Sections; besides, there is an addendum. Section I treats of the general bicursal bicircular quartic and reckons with the inter-relations between the systems of circles of double contact and the tetrads of focal conics and of circles of inversion. Then Section II is devoted to the consideration of the special type of bicircular quartic, having one or more points of undulation. Section III touches very briefly on the 'polyzomal' character of the curve, whereas Section IV deals with the degenerate variety viz., a bicursal circular cubic. Finally the addendum contains a few additional propositions.

Although the subject-matter is classical in origin and its antiquity dates back to the days of savants like Casey and Cayley, still my honest conviction is that a very decent amount of what appears in this paper is original with me. Even in the disposal of known results, I believe that either a sort of rapprochement has been effected between them, or else a new line of argument has been adopted.

The subject has been mainly discussed from a *projective* stand-point, and its mode of analysis has been made as elementary as possible, having nothing to do with elliptic functions. Other papers, which have a bearing on the subject and are almost nearing completion will, I hope, see the light of day in the near future.

SECTION I

CERTAIN NOVEL PROPERTIES OF THE GENERAL (BICURSAL) BICIRCULAR QUARTIC

1. As shewn by Dr. Casey, a bicursal bicircular quartic Γ possesses four *'circles of inversion'* $(\Pi, \Pi_1, \Pi_2, \Pi_3)$ and four ordered *'focal conics'* $(\Sigma, \Sigma_1, \Sigma_2, \Sigma_3)$, related to one another in such a way that, if (Π_r, Σ_r) be any of the four circle-conic pairs, the quartic Γ is derivable as the envelope of the ∞^1 circles, whose centres move on Σ_r and which cut

Π_r orthogonally. For obvious reasons, the four families of circles are alternatively designable as the (four) determinate sets of circles, that can be drawn so as to have double contacts with the given bicircular quartic Γ; the subsidiary property that the chord of contact of any of the circles with Γ should pass through one or other of four *fixed* points *viz.*, D, A, B, C is a mere consequence of the fact of double contact; it is scarcely necessary to add that the four points last mentioned are respectively the centres of the four circles Π, Π_1, Π_2, Π_3 and are often called *'centres of inversion'* of Γ. It is easy to see that each of the four systems of circles (of double contact) embraces

(*a*) two and only two special circles of *infinite* radii, which are none other than (two) bitangents of Γ;

(*b*) four and only four special circles of *zero* radii, whose centres are (Plücker's) foci of Γ;

(*c*) four and only four special circles of *four-pointic* contact with Γ, their points of contact being *cyclic* points (of Γ).

In other words, closely linked with the four circle conic pairs *viz.*,

$$(\Pi, \Sigma), \quad (\Pi_1, \Sigma_1), \quad (\Pi_2, \Sigma_2) \text{ and } (\Pi_3, \Sigma_3), \qquad \text{(I)}$$

there exist (i) four sets of circles of double contact,
(ii) four pairs of bitangents,
(iii) four tetrads of foci,
and (iv) four tetrads of cyclic points and four (associated) tetrads of circles of four pointic contact.

For felicity of expression we shall use the adjunct *'cognate'* in a special sense. Thus two or more circles of double contact, which belong to the *same* family and therefore answer to the *same* circle-conic pair (Π_r, Σ_r), will be spoken of as mutually *cognate*; the chords of contact of such cognate circles will, of course, pass through a centre of inversion, *viz.*, the centre of the related circle Π_r. Similarly, two bitangents, related to the *same* circle-conic pair (Π_r, Σ_r), will be termed *cognate*, and two foci or two cyclic points or two circles of four-pointic contact will be said to be *cognate* or *non-cognate* according as they do or do not belong to the family of circles, initiated by the *same* circle-conic pair.

The above nomenclature being adopted, we may declare at once that a bicursal bicircular quartic Γ possesses, in general,

(i) four distinct sets of ∞^1 *cognate* circles of double contact;
(ii) four distinct pairs of *cognate* bitangents;
(iii) four distinct tetrads of *cognate* foci;
and (iv) four distinct tetrads of *cognate* cyclic points and also of (associated) *cognate* four-pointic circles.

For the purpose of investigating the mutual relations subsisting among two or more *cognate* entities (*e.g.*, circles of double contact or bitangents or foci or cyclic points or four-pointic circles), it suffices to concentrate our attention upon *any one* of the four systems, defined by the four circle-conic pairs marked (I); for the intrinsic geometrical properties, holding for any one of them, must hold likewise for each of the remaining three.

To that end, we shall adopt an indirect line of argument and shall, in the next few articles (2 – 5), concern ourselves with only one of the four families of circles of double contact.

2. At the very outset we shall make use of a lemma that, if L and M be the two *fixed* points (real or imaginary), where a *given* circle S is met by a *given* right line T, the ***most general*** equation of a bicircular quartic, having double contact with S at L, M, may be written as

$$SS' = T^2, \tag{1}$$

where S' is the ***most general*** expression of a circle, involving necessarily *four** disposable or arbitrary parameters. Consequently, if a given bicircular quartic Γ have S for a circle of double contact and T for the related chord of contact, its symbolic equation must be of the form (1), where S is a perfectly determinate (and determinable) expression of a circle. An important inference can be drawn from this. For the very form of (1) signifies that S' is also a (cognate) circle of double contact, whose chord of contact is located along the self-same line T. Thus *with every circle S, having double contact with a given bicircular quartic, there is associated another uniquely determinate circle S' (of double contact), having precisely the same line of contact. Pairs of circles (S, S')* of double contact—related to each other in the afore-said manner—will be said to be mutually ***correlated.***

Before we take to Cartesian analysis, we have to make a special choice of the origin. Suppose that (L', M') are the two points of double contact of S', (L, M) being, as before the points of contact of S. Mark the centre O of the involution-system (determined uniquely on the line of contact T) by the two pairs of points (L, M) and (L', M'), so that

$$OL.OM = OL'.OM'. \tag{2}$$

If, as a matter of expediency, we take O as the origin of rectangular Cartesian coordinates and the (common) line of double contact, *viz.*, T as the axis $(x = 0)$, we may set

$$T \equiv 2kx, \tag{3}$$

$$S \equiv x^2 + y^2 + 2gx + 2fy + c, \tag{4}$$

and

$$S' \equiv x^2 + y^2 + 2g'x + 2f'y + c, \tag{5}$$

where, of course, g, f, g', f', c, k are all constants. Evidently the absolute terms in S and S' are identical by virtue of the equality (2).

The bicircular quartic Γ being, as before, given by the equation (1), it is certainly the envelope of the series of circles (of double contact), *viz.*,

$$S + 2\lambda T + \lambda^2 S' = 0, \tag{6}$$

where λ is a *variable* parameter. Plainly for any particular value of λ, the centre of (6) is the point

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* Manifestly the general expression of a circle (e.g., S) involves *four* arbitrary constants, although the general equation contains only *three* such constants

$$\left(-\frac{g+2\lambda k+\lambda^2 g'}{1+\lambda^2},\quad -\frac{f+\lambda^2 f'}{1+\lambda^2}\right), \tag{7}$$

and its radius

$$=\frac{\sqrt{\{(g+2\lambda k+\lambda^2 g')^2+(f+\lambda^2 f')^2-c(1+\lambda^2)^2\}}}{1+\lambda^2}. \tag{8}$$

Manifestly, when λ varies, the centre-locus is the conic Σ, given by

$$[(f-f')x-(g-g')y+(fg'-f'g)]^2+4k^2(y+f)(y+f')=0. \tag{9}$$

Further the Jacobian of any three circles $(\lambda_1, \lambda_2, \lambda_3)$ of the system (6), being the same as that of (S, S', T), is readily found to be no other than a circle (say, Π)*, having for its equation

$$x^2+y^2=c. \tag{10}$$

The irresistible conclusion is, then, that the family (6) of circles of double contact (including within its fold the correlated circles S, S') have their centres situated on the conic Σ and intersect the Jacobian circle Π orthogonally. To borrow Dr. Casey's phraseology, Σ must be called the *focal conic* and Π the associated *circle of inversion*, appertaining to the correlated circles of double contact S, S'. To put it otherwise, (6) defines one among four distinct families of circles of double contact, the related circle-conic pair being (Π, Σ).

Equating the radius (8) of the circle (6) to zero, we promptly perceive that the four 'cognate' foci of Γ, that lie on Π, have for their coordinates:

$$\left(-\frac{g+2\mu k+\mu^2 g'}{1+\mu^2},\quad -\frac{f+\mu^2 f'}{1+\mu^2}\right),\quad (\mu=\mu_1, \mu_2, \mu_3, \mu_4), \tag{11}$$

it being premised that $\mu_1, \mu_2, \mu_3, \mu_4$ are the roots of the biquadratic in μ, *viz.*,

$$(g+2\mu k+\mu^2 g')^2+(f+\mu^2 f')^2-c(1+\mu^2)^2=0. \tag{12}$$

When, however, we look for the four-pointic circles, included in the system (6), we have to impose the condition that the two points of contact of (6) with the envelope should coincide. Obviously this requires the two circles

$$S+\lambda T=0 \quad \text{and} \quad T+\lambda S'=0$$

to touch each other, so that one (and therefore both) of them must touch their radical axis *viz.*

$$S-S'+(\lambda-1/\lambda)T=0.$$

The condition of contact is finally found to be

$$\begin{aligned}(f^2-c)\{k.(\lambda^2-1)+\lambda(g-g')\}^2+\lambda^2.(f-f')^2(g+\lambda k)^2\\ -c\lambda^2(f-f')^2-2f\lambda(f-f')(g+\lambda k)\{k(\lambda^2-1)+\lambda(g-g')\}=0\cdot\end{aligned} \tag{13}$$

Hence if $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be the four roots of this biquadratic in λ, the four 'cognate' cyclic points of Γ—that are situated on the circle of inversion Π—have for their Cartesian coordinates:

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* It can be directly verified that S, S', T being defined by (4), (5), (3), the (bicircular) quartic Γ, denoted by (1), inverts into itself *w. r. t.* the auxiliary circle (Π), *viz.*, (10).

$$\left[\frac{c(f-f')(1+\lambda^2)}{\Omega}, \quad -\frac{c(1+\lambda^2)\{(g-g')+k(\lambda-1/\lambda)\}}{\Omega}\right], \tag{14}$$

where

$$\Omega \equiv (f+\lambda^2 f')\{(g-g')+k(\lambda-1/\lambda)\}-(f-f')(g+2\lambda k+\lambda^2 g')$$

and $\lambda \equiv \lambda_1, \lambda_2, \lambda_3, \lambda_4$ in succession.

Furthermore the tangents to Γ at the four cyclic points—which are alternatively designable as tangents, drawn to Γ from the centre of inversion O—are given by

$$\{(g-g')+k(\lambda-1/\lambda)\}x+(f-f')y = 0, \tag{15}$$

whereas their osculating circles—having *four-pointic* contacts with Γ—are given by

$$(1+\lambda^2)(x^2+y^2+c)+2(g+2\lambda k+\lambda^2 g')x+2(f+\lambda^2 f')y = 0, \tag{16}$$

it being postulated in both (15) and (16), that λ is to take on the values $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ in succession.

Thus we have taken account of the tetrads of (i) '*cognate*' foci, (ii) '*cognate*' cyclic points, and (iii) '*cognate*' four-pointic circles, that are intimately related to the circle-conic pair (Π, Σ).

It may be remarked in passing that the *bicursal* bicircular quartic Γ being of class eight, it must be possible to draw from the centre of inversion (O), $4(=8-4)$ tangents (to Γ) other than the four lines (15), (whose points of contact are the four cyclic points lying on Π). These *four extra* tangents (form O) will be reckoned with in the next article.

3. Retaining the conventions and notations of Art. 2, we may represent the given (bicursal) bicircular quartic in the form

$$SS' = T^2, \tag{1}$$

where

$$S \equiv x^2+y^2+2gx+2fy+c, \quad S' \equiv x^2+y^2+2g'x+2f'y+c \text{ and } T \equiv 2kx. \tag{2}$$

Plainly (1) can, by Elementary Algebra, be transformed into

$$UV = W^2, \tag{3}$$

where $U = 0$ and $V = 0$ are the *separate* equations of the two right lines, whose *combined* equation is

$$\{(g-g')^2+4k^2\}x^2+2(f-f')(g-g')xy+(f-f')^2y^2 = 0 \tag{4}$$

and

$$W \equiv x^2+y^2+(g+g')x+(f+f')y+c = 0 \tag{5}$$

is a circle

Clearly the equation (3), interpreted geometrically, means that each of the two right lines $U = 0$, $V = 0$ is a bitangent (or double tangent) of the curve Γ, and that their two pairs of points of contact are *concyclic*, situated, as they are, on the circle $W = 0$.

We are now in a position to account for the four 'extra' tangents (from O), as anticipated at the end of the previous article. For each of the two bitangents takes the place of two *ordinary tangents*, that can be drawn from the centre of inversion O.

Accordingly, the aggregate number of tangents from O comes up to $8(=4$ due to the tangents at the four cyclic points $+4$ due to the two bitangents). This disposes of the query suggested in the foregoing article.

It is worthy of notice that the centre K of the circle W, *viz.*,

$$\{-\tfrac{1}{2}(g+g'),\quad -\tfrac{1}{2}(f+f')\} \tag{6}$$

is the mid-point of the line joining the centres $(-g, -f)$ and $(-g', -f')$ of the circles S, S'. It is rather a tame affair to verify another interesting property of the point K, *viz.*, that it is the centre of the focal conic Σ, whose equation marked (9) in Art. 2, is

$$[(f-f')x-(g-g')y+(fg'-f'g)]^2+4k^2(y+f)(y+f')=0.$$

Remembering that the four focal conics $\Sigma, \Sigma_1, \Sigma_2, \Sigma_3$ of a bicursal bicircular quartic possess a common centre *viz.* K, we are entitled to finalise our conclusions in the form of two theorems, *viz.*,

Theorem I. *If S, S' be two correlated circles of double contact of a bicursal bicircular quartic Γ, the line, joining their centres, has, for its middle point, the common centre K of the four focal conics of Γ.*

Theorem II. *The four points of contact of any two cognate bitangents of a bicursal bicircular quartic lie on a circle, whose centre is precisely the same point K (viz., the common centre of the four focal conics).*

4. Adhering to the notations and conventions of Arts. 2 and 3, we shall now suppose that the circle S given by

$$S \equiv x^2+y^2+2gx+2fy+c=0 \tag{1}$$

is a *four-pointic* circle, (*i.e.*, a *special* circle, having two coincident points of contact), and that

$$S' \equiv x^2+y^2+2g'x+2f'y+c=0 \tag{2}$$

is the correlated circle of double contact.

As before, the bicircular quartic Γ is

$$SS'=T^2, \tag{3}$$

where

$$T \equiv 2kx.$$

Since the two points of double contact of S with Γ must coincide at the cyclic point, (say, α), it follows that the chord of contact (*viz.*, $T=0$, *i.e.*, $x=0$) must touch the circle $S=0$ at α.

Consequently we must have

$$f^2=c; \tag{4}$$

subject to this condition, the coordinates of the cyclic point α are $(0, -f)$, and its centre of curvature (say, α'), which is none else than the centre of S, is $(-g, -f)$.

As is to be expected from other considerations, the cyclic point $\alpha\,(0, -f)$ must, as a consequence of (4), lie on the circle of inversion Π, *viz.*,

$$x^2+y^2=c. \tag{5}$$

Manifestly, the line $\alpha\alpha'$, whose equation is

$$y+f=0 \tag{6}$$

touches the focal conic Σ, whose equation is, by Art. 2,

$$[(f-f')x-(g-g')y+(fg'-f'g)]^2+4k^2(y+f)(y+f')=0, \tag{7}$$

the actual point of contact being the point α' $(-g, -f)$.

The adjoining diagram shows the mutual configurations of the special points α, α' and the circles Π, S and the conic Σ.

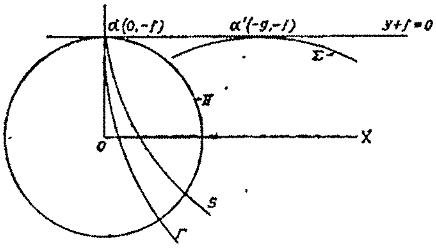

FIG. 1

It is now crystal-clear that the line $\alpha\alpha'$, which is normal to the quartic Γ at α, may also be characterised as a common tangent to the circle of inversion Π and the focal conic Σ, defined respectively by the Cartesian equations (5) and (7).

From considerations of symmetry, we infer at once that, if β, γ, δ be the three cyclic points (of Γ) cognate with α, so that α, β, γ, δ make up the tetrad of points of intersection of the bicircular quartic Γ with the circle of inversion Π, and if β', γ', δ' be the centres of curvature answering respectively to β, γ, δ, then not only α' but β', γ', δ' will also lie on the focal conic Σ, and at the same time the four lines $\alpha\alpha'$; $\beta\beta'$, $\gamma\gamma'$, $\delta\delta'$ will be the (four) common tangents of the circle-conic pair (Π, Σ). The second figure shows the relative positions of the afore-said set of points.

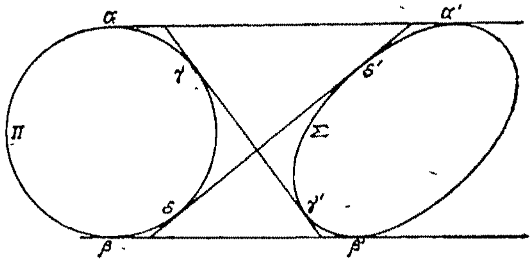

FIG. 2

Thus *whereas the four 'cognate' foci of the bicursal bicircular quartic* Γ, *that appertain to the circle of inversion* Π, *are the four points of intersection of* Π *with the related focal conic* Σ, *the four 'cognate' cyclic points* (α, β, γ, δ), *that appertain to* Π, *are the (four) points of contact of* Π *with the (four) tangents, which it* (Π) *has in common* *with Σ; moreover the other ends* $(\alpha', \beta', \gamma', \delta')$ *of the said common tangents are respectively the centres of curvature of* $(\alpha, \beta, \gamma, \delta)$.

It is palpably plain that similar properties must hold for the other three tetrads of 'cognate' cyclic points, attaching respectively to the three circle-conic pairs: (Π_1, Σ_1), (Π_2, Σ_2) and (Π_3, Σ_3).

The above result can, in any particular case, be made use of to ascertain the *reality* or otherwise of the cyclic points of any of the four 'cognate' tetrads, (say $\alpha, \beta, \gamma, \delta$). For if α is to be a *real* cyclic point of Γ, its centre of curvature α' (situated on Σ)—and along with it the common tangent $\alpha\alpha'$—must be *real*. Consequently the number of *real* cyclic points of the 'cognate' tetrad $(\alpha, \beta, \gamma, \delta)$ is precisely the same as that of *real* common tangents of Π and Σ Evidently this is the analogue of the corresponding theorem on foci, *viz.*, that the number of *real* foci of the 'cognate' tetrad, attaching to a circle of inversion Π, is just equal to the number of *real* points of intersection of Π and Σ. It is superfluous to add that, for a *real* bicircular quartic, the integers (N and N'), representing the numbers of *real* cyclic points and of *real* foci, belonging to the same cognate tetrad, must be *even*; the connection between N and N' can be easily traced in any particular case. Thus, for instance when Π and Σ are wholly external to each other, $N = 4$ and $N' = 0$, and so on.

Returning now to any of the four tetrads of cognate cyclic points (say $\alpha, \beta, \gamma, \delta$), as shown in Fig. 2, and attending to Roberts's theorem[†], *viz.*, that the algebraic sum of the reciprocals of the common tangents of a circle and an algebraic (plane) curve is *nil*, we immediately derive the relation:

$$\frac{1}{\alpha\alpha'} \pm \frac{1}{\beta\beta'} \pm \frac{1}{\gamma\gamma'} \pm \frac{1}{\delta\delta'} = 0.$$

Inasmuch as the radii of curvature $\rho_1, \rho_2, \rho_3, \rho_4$ of Γ at the points $\alpha, \beta, \gamma, \delta$ are respectively equal to $\alpha\alpha', \beta\beta', \gamma\gamma', \delta\delta'$, we at once arrive at the following proposition:

If $\rho_1, \rho_2, \rho_3, \rho_4$ *be the radii of curvature of a bicursal bicircular quartic at the four cyclic points of any of the four cognate tetrads, then*[*]

$$\frac{1}{\rho_1} \pm \frac{1}{\rho_2} \pm \frac{1}{\rho_3} \pm \frac{1}{\rho_4} = 0.$$

An alternative enunciation runs thus:

The algebraic sum of the curvatures of a bicursal bicircular quartic at any four cognate cyclic points is nil.

5. We shall now talk a little about the evolute of a bicursal bicircular quartic.

In the first place we remark that, if α be a cyclic point on any given curve Γ—which may or may not be a bicircular quartic,—and if α' be the centre of curvature of Γ at α, then α' must be a cusp on the evolute (Ω), the associated (cuspidal) tangent being none other than the normal $\alpha\alpha'$ of the original curve Γ at α. When we apply this principle to a bicursal bicircular quartic Γ, we find, at once, on reference to Fig. 2 of Art 4, that the normal to Γ at the cyclic point α touches both the curves (*viz.*, Σ and Ω) at one and the same point α', this latter point being a cusp on Ω; thus the line $\alpha\alpha'$, which is an *ordinary* tangent to Σ at α', is but the *cuspidal* tangent to Ω at the (cuspidal) point α'. For obvious reasons, similar relations hold for the other three pairs of points (β, β'), (γ, γ') and (δ, δ'). We may therefore summarise our conclusions in various forms:

[†] Cf R. A. Roberts's 'Examples and Problems on Conics and Cubics', (1882), Ex. 387 (P. 181).

[*] Needless to say, the algebraic signs (+, −) are to be selected in a particular or definite manner in order that the relation quoted above may be valid.

Theorem I. *The evolute Ω of a bicursal bicircular quartic Γ possesses, in general, sixteen cusps (real or imaginary), which lie, four by four, on the four focal conics $\Sigma, \Sigma_1, \Sigma_2, \Sigma_3$. Furthermore Ω touches each of these conics at the four cusps, lying on its periphery, the (four) cuspidal tangents (to Ω) being ordinary tangents to the conic as also to the related circle of inversion and normals to the quartic Γ at the four 'cognate' cyclic points, attaching thereto.*

Theorem II. *If, for a given bicursal bicircular quartic Γ, $(\Pi, \Pi_1, \Pi_2, \Pi_3)$ be the four circles of inversion and $(\Sigma, \Sigma_1, \Sigma_2, \Sigma_3)$ be the four corresponding focal conics, the four tetrads of common tangents (real or imaginary) of the four circle-conic pairs* (Π, Σ), (Π_1, Σ_1), (Π_2, Σ_2), (Π_3, Σ_3) *are simply the sixteen cuspidal tangents of the evolute, the corresponding cusps being seated, four by four, on the conics $\Sigma, \Sigma_1, \Sigma_2, \Sigma_3$.*

6. Before we proceed further, it is worth while to consider an *alternative* method† of investigating the common tangents of a circle of inversion Π and its related focal conic Σ.

If Π and Σ* be taken in the respective Cartesian forms:

$$(x-f)^2+(y-g)^2 = k^2 \text{ and } \frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1, \tag{1}$$

the equation of the bicircular quartic is easily seen to be

$$(x^2+y^2+c^2)^2 = 4\{a^2(x-f)^2 \pm b^2(y-g)^2\} \quad \text{where } c^2 = k^2-f^2-g^2. \tag{2}$$

An *arbitrary* point P on Π being taken as

$$(f+k\cos t,\ g+k\sin t), \tag{3}$$

(where the parameter t is *undefined*), the tangent to Π at P is

$$x\cos t+y\sin t = p, \text{ where } p \equiv f\cos t+g\sin t+k.$$

Imposing the condition that this line may touch the conic Σ, we get

$$\left.\begin{aligned} p^2 &= a^2\cos^2 t \pm b^2\sin^2 t, \\ \text{i.e.,}\quad (f\cos t+g\sin t+k)^2 &= a^2\cos^2 t \pm b^2\sin^2 t. \end{aligned}\right\} \tag{4}$$

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† Cf. Hilton's *Plane Algebraic Curves* (1920). Ex. 15 (P. 309) and Ex. 6 (P223), wherein an *indirect* reference to the common tangents is given.

* For a *non-degenerate* bicircular quartic, the focal conics must be *central* (i e , elliptic or hyperbolic). The *degenerate* type of bicircular quartic (consisting of the line at infinity and a circular cubic) and having necessarily *parabolic* focal conics, is here left out of consideration.

This biquadratic equation in $\tan t/2$ indirectly determines the four points of contact of the circle Π with the (four) tangents, which it has in common with Σ.

Observing that (4) is also the condition that the point P, defined by (3), may lie on the curve Γ, we infer that the four points of intersection of the circle Π with the bicircular quartic Γ are precisely the same as the points of contact (with Π) of the four common tangents of Π and Σ.

Hence bearing in mind that the four points of intersection of a bicircular quartic Γ with a circle of inversion are but cyclic points of Γ, we are once again led to the theorem—proved in Art. 5—*viz.*, that the 16 cyclic points of a bicursal bicircular quartic are simply the points of contact of the four circles of inversion Π, Π_1, Π_2, Π_3, with the tetrads of tangents, which they have in common with their associated focal conics Σ, Σ_1, Σ_2, Σ_3.

SECTION II

BICIRCULAR QUARTICS WITH ONE OR MORE POINTS OF UNDULATION

7. It is common knowledge that a plane curve, given by its *point-equation*, does not normally possess any point of undulation; as a matter of fact, the possession of a point of undulation entails a sort of limitation on the nature of the curve. Plain reasoning shows that a point of undulation, when existent on a curve, may be regarded as a *special* type of cyclic point, the *specialty* consisting in that the circle, which passes through the four consecutive points, is of *infinite dimensions* so as to approximate to the form of a *straight line*. It follows automatically that, when a cyclic point 'α' of a curve reduces to a point of undulation, its centre of curvature must move off to infinity so that the normal at 'α' becomes an asymptote of the evolute of the given curve.

Let us now proceed to discuss the possible existence of a point of undulation on a bicursal bicircular quartic Γ. If we now refer to Fig. 2. of Art 4, the fact stands out that the cyclic point α will degenerate into a point of undulation on Γ, if and only if its centre of curvature, *viz.*, the point α' (lying on Σ) be at infinity. That is to say, α will be a point of undulation of Γ if the tangent to Π at α be an asymptote of Σ, and conversely. Accordingly, ***the necessary and sufficient condition for the bicircular quartic Γ to have a point of undulation is that an asymptote of one of the focal conics (say, Σ) should touch the associated circle of inversion Π; furthermore when this condition is fulfilled, the actual point of contact of the said asymptote with Π will be the point of undulation.*** Palpably the conditions for two or more points of undulation can be obtained by mere superposition of the aforesaid conditions. Thus ***the conditions—at once necessary and sufficient—for a bicursal bicircular quartic to have two 'cognate'* points of undulation are that both the asymptotes of one of the focal conics should touch the corresponding circle of inversion.*** Inasmuch as the four focal conics Σ, Σ_1, Σ_2, Σ_3 have altogether 4×2 or 8 asymptotes (real or imaginary), it follows that the total number of points of undulation (on a bicursal bicircular quartic) can never exceed 8. Further consideration shows that the highest number (of undulations) cannot even be as large as 8. For a bicursal bicircular quartic has ordinarily 12 points of inflexion and cannot therefore have more than 6(= ½ of 12) undulations, the reason being that a point of undulation is formed by the union of *two* points of inflexion. Inquisitive students may propose to ascertain the precise number of undulations, that a bicircular quartic can have under the most auspicious conditions.

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* Points of undulation are here termed '*cognate*' in the sense that they are situated on the *same* circle of inversion.

8. Leaving aside theoretical speculations, we shall now obtain the Cartesian equations of a bicircular quartic, actually possessing one or more points of undulation.

As in Art 6 we may initially represent a bicircular quartic in the form:

$$(x^2+y^2+c^2)^2 = 4\{a^2(x-f)^2-b^2(y-g)^2\}, \quad (c^2 = k^2-\alpha^2-\beta^2),$$

it being supposed that a circle of inversion Π and its associated focal conic Σ are given by

$$(x-f)^2+(y-g)^2 = k^2$$

and

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Expressing the condition that Π may touch one (or both) of the lines $y = \pm(b/a).x$, we are squarely led to the following special results:

(*i*) that the bicircular quartic, defined by

$$\left\{x^2+y^2-\frac{(af+bg)^2}{a^2+b^2}\right\}^2 = 4\{a^2(x-f)^2-b^2(y-g)^2\}$$

has the point

$$\left\{\frac{a(af+bg)}{a^2+b^2}, \quad \frac{b(af+bg)}{a^2+b^2}\right\}$$

for a point of undulation, the tangent thereat being

$$ax+by = af+bg;$$

(*ii*) that the bicircular quartic, defined by

$$\left\{x^2+y^2-\frac{a^2f^2}{a^2+b^2}\right\}^2 = 4\{a^2(x-f)^2-b^2y^2\}$$

has two points of undulation, *viz.*,

$$\left(\frac{a^2f}{a^2+b^2}, \frac{abf}{a^2+b^2}\right) \quad \text{and} \quad \left(\frac{a^2f}{a^2+b^2}, -\frac{abf}{a^2+b^2}\right),$$

the tangents at which are respectively

$$a(x-f)+by = 0 \quad \text{and} \quad a(x-f)-by = 0;$$

(*iii*) that the bicircular quartic, defined by

$$\left\{x^2+y^2-\frac{b^2g^2}{a^2+b^2}\right\}^2 = 4\{a^2x^2-b^2(y-g)^2\}$$

has two points of undulation, *viz.*,

$$\left(\frac{abg}{a^2+b^2},\ \frac{b^2g}{a^2+b^2}\right) \text{ and } \left(-\frac{abg}{a^2+b^2},\ \frac{b^2g}{a^2+b^2}\right),$$

the tangents at which are respectively

$$ax+b(y-g)=0 \quad \text{and} \quad ax-b(y-g)=0.$$

Arguing from first *principles*, we can easily write down the *general* equation of an extensive class of bicircular quartics, having one or more points of undulation. It is easy to see that, *if* $U=0$ *and* $V=0$ *be a pair of right lines, one of which (say, the first) touches the circle* $S=0$ *at a point (say,* P*), the equation*:

$$S^2 = UV \tag{A}$$

must denote a bicircular quartic, of which P *will be a point of undulation, the tangent at this undulation being the line* $U=0$.

It is hardly necessary to add that, *when both the right lines* $U=0$, $V=0$ *touch the circle* $S=0$, *the respective points of contact being* P *and* Q, *the symbolic equation* (A) *will denote a bicircular quartic, having both* P *and* Q *as points of undulation.*

As numerical illustrations of the above proposition, we may mention that of the two bicircular quartics:

$$(x^2+y^2-3x+10y-15)^2 = (5x-12y-152)(x+y+1)$$

and

$$(x^2+y^2-2x+4y-12)^2 = (4x+y-19)(4x-y+11),$$

the first has the single undulation $(4, -11)$, whereas the second has the pair of undulations $(5, -1)$ and $(-3, -1)$. Any number of (bicircular) quartics of this category can be obtained if one utilises the symbolic equation (A).

9. We will now make a few general observations on the *generation* of bicircular quartics, endowed with one or more points of undulation. The cardinal point to be noted in this connection is that although the *general* type of bicircular quartic Γ (bicursal or unicursal) may *not*, (and, in general, will *not*) possess even a single point of undulation, still it can be made to generate, *by process of inversion*, a *special* type of bicircular quartic, possessing one or two points of undulation. For during (circular) inversion, a cyclic point P on any given curve Γ—not necessarily a bicircular quartic—will be converted into a cyclic point or a point of undulation on the inverse curve, according as the origin of inversion does not or does lie on the osculating circle of Γ at P. This general principle being borne in mind, we learn that, if the origin of inversion O be taken somewhere on the osculating circle of a bicircular quartic Γ at a cyclic point P, the inverse curve will be another bicircular quartic having a point of undulation (*viz.*, the inverse of P). In particular, when the origin of inversion is taken at a *common* point of two four-pointic circles of a bicircular quartic, the inverse curve—which is also a bicircular quartic—will have *two* points of undulation.

SECTION III

SYMBOLIC IRRATIONAL REPRESENTATION OF A BICIRCULAR QUARTIC

10. Reverting to the topic of the *general* bicursal bicircular quartic Γ, we now proceed to establish one of its irrational symbolic equations. From Art. 2 we know that, if $S = 0$ and $S' = 0$ denote a pair of 'correlated' circles (of double contact)—belonging naturally to the *same* family,—and if $T = 0$ denote their common line of contact, the equation can be thrown into the form

$$SS' = T^2. \tag{1}$$

This can at pleasure be put in the form:

$$(S+2\lambda T+\lambda^2S')S' = (T+\lambda S')^2, \tag{2}$$

shewing that, for different values of the parameter λ, the equation

$$S+2\lambda T+\lambda^2S' = 0 \tag{3}$$

will denote the set of 'cognate' circles of double contact (that include S, S'), and the equation:

$$T+\lambda S' = 0$$

will denote the corresponding line of contact.

If we now pick out, at random, three arbitrary circles of the system (3), by assigning in succession three arbitrary values λ_1, λ_2, λ_3 to λ, the equation (2) can be modified into each of the three equivalent forms.

$$S_1S' = T_1^2, \quad S_2S' = T_2^2 \text{ and } S_3S' = T_3^2, \tag{4}$$

where

$$S_r \equiv S+2\lambda_rT+\lambda_r^2S' \text{ and } T_r \equiv T+\lambda_rS', \quad (r = 1, 2, 3).$$

Because the three *concurrent* lines T_1, T_2, T_3 must conform to a certain linear relation of the form:

$$pT_1+qT_2+rT_3 = 0,$$

(where p, q, r are constants depending on λ_1, λ_2, λ_3), we infer immediately from the three equations, marked (4), that the equation of Γ admits of the irrational form.

$$\sqrt{(lS_1)}+\sqrt{(mS_2)}+\sqrt{(nS_3)} = 0, \tag{5}$$

where l, m, n are constants*. Thus the *general* bicircular quartic must be expressible in the symbolic irrational form (5), it being postulated that S_1, S_2, S_3 are circles of double contact, belonging to the *same* 'cognate' family. One or more of these circles may, as a special case, be four-pointic circles or point-circles (at the foci). When, however, rectangular Cartesian coordinates are used, and the common coefficient of x^2, y^2 in S_1, S_2, S_3 is unity, the bicircular quartic (5) will break up into a circular cubic and the line at infinity under the condition:

$$\sqrt{l}\pm\sqrt{m}\pm\sqrt{n} = 0.$$

Plainly the very form of the equation (5) reveals that a bicircular quartic (as well as a circular cubic) belong to the class of curves, called *'polyzomal'* by Professor Cayley.

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* Obviously, l, m, n = p^2, q^2, r^2 respectively.

SECTION IV

CIRCULAR CUBICS

11. Inasmuch as the general type of circular cubic, when coupled with the line at infinity, constitutes a degenerate (or improper) bicircular quartic, it follows that almost every property of a bicircular quartic must have its *counterpart* in the theory of circular cubics. Of course, a property of a (bicircular) quartic must have to be *suitably* modified so as to be applicable to a (circular) cubic. In fact, the theory of circular cubics is very often sub-ordinated to that of bicircular quartics. However, there is no bar to pursuing an independent line of investigation, in which a circular cubic is studied on its own merits without any reference to bicircular quartics. In this section we intend to touch briefly on an ***independent*** treatment of the circular cubic.

As is well-known, a bicursal circular cubic possesses, in much the same way as a bicursal bicircular quartic does, four distinct families of circles of double contact. A little reflection shows that, if $S = 0$ be any such circle (of double contact) and $u = 0$ be its chord of contact, the equation of the (circular) cubic Γ must be amenable to the form:

$$S.v = u^2, \tag{1}$$

where $v = 0$ is a certain right line. A glance at (1) reveals the fact that the point O (given by $u = 0$, $v = 0$) lies on the curve Γ and has, for its tangent, the line $v = 0$, which is parallel to the *real* asymptote of Γ. The point O is accordingly a *centre of inversion*; it goes without saying that there must be altogether *four* such centres (of inversion), seeing that there must exist four tangents (to Γ), which are parallel to the *real* asymptote. Manifestly, then, the equation (1) can be made use of to study the properties of any one of the four families of 'cognate' circles of double contact, attaching respectively to the four centres of inversion. To be more precise, we choose at random one of the four centres of inversion (say, O), and take it as the origin of Cartesian coordinates (in general, oblique).

At the very outset we may set

$$u = x \text{ and } v = y,$$

so that the coordinate axis $y = 0$ is parallel to the real asymptote of Γ; we may further write

$$S \equiv a(x^2+2xy\cos\omega+y^2)+2gx+2fy+c, \tag{2}$$

where ω stands for the obliquity. We shall now make a special choice of the circle of double contact, the speciality consisting in that the two attached points of contact shall be coincident. For this to be possible the line $x = 0$ must touch the circle $S = 0$, leading to the condition

$$f^2 = ca. \tag{3}$$

Subject to this condition, $S = 0$ will be the osculating (*i.e.*, four-pointic) circle (of Γ) at the cyclic point α, *viz.*, $(0, -f/a)$, and the centre of curvature (α') for α—which is none other than the centre of S—is designable as the point of intersection of the line-pair:

$$a(x\cos\omega+y)+f=0, \tag{4}$$

and

$$a(x+y\cos\omega)+g=0. \tag{5}$$

The Cartesian equation (1) of Γ now becomes

$$S.y=x^2,$$

which can be turned into

$$(S+2\lambda x+\lambda^2 y)y=(x+\lambda y)^2, \tag{6}$$

(where λ is a parameter). It is evident that, for *different* values of λ, the equation

$$S+2\lambda x+\lambda^2 y=0 \tag{7}$$

defines a set of ∞^1 circles of double contact, whereas the equation

$$x+\lambda y=0 \tag{8}$$

determines their chords of contact, (which clearly form a pencil through the related centre of inversion O).

The Jacobian of three arbitrary circles (with λ_1, λ_2, λ_3 as parameters) being the *invariable* circle (Π), *viz*:

$$x^2+2xy\cos\omega+y^2=c/a, \tag{9}$$

it follows that the series of circles (7) intersect Π at right angles.

Moreover Chasles' characteristics (μ, ν) for (7) being given by

$$\mu=2 \quad\text{and}\quad \nu=4,$$

the centre-locus (say, Σ) can be legitimately expected to be a curve of the second degree This locus Σ can without difficulty be verified to be the parabola:

$$[a(x+y\cos\omega)+g]^2=-2[a(x\cos\omega+y)+f]. \tag{10}$$

Hence, using Casey's nomenclature we can affirm that the circle Π, as given by (9) is one of the four 'circles of inversion', and that Σ, given by (10), is the related 'focal parabola', so that the original cubic Γ, *viz.*, (6) is definable as the envelope of circles, whose centres lie on Σ, and which cut Π orthogonally.

Plainly the cyclic points and foci, lying on Π, can be found in much the same way as in the case of a bicircular quartic (Art 2).

Thus imposing the condition that (7) shall denote a point-circle, (so that the L. S. shall break up into two linear factors), we find the coordinates of the four foci (attaching to Π and therefore lying on its periphery) to be

$$\left(\frac{\mu^2\cos\omega-2\mu+2f\cos\omega-2g}{2a\sin^2\omega},\quad -\frac{\mu^2-2\mu\cos\omega-2g\cos\omega+2f}{2a\sin^2\omega}\right),$$

where $\mu=\mu_1, \mu_2, \mu_3, \mu_4$ (successively), these being the four roots of the biquadratic in μ, *viz.*,

$$\begin{vmatrix} a, & a\cos\omega, & g+\mu \\ a\cos\omega, & a, & f+\mu^2/2 \\ g+\mu, & f+\mu^2/2, & c \end{vmatrix}=0.$$

When, however, we introduce the condition that the line (6) shall touch the circle (7), we immediately obtain the four cyclic points situated on Π, *viz.*,

$$(0, \ -f/a)$$

and

$$\left\{-\frac{\lambda(\lambda^2+2\lambda g-2f)}{2a(\lambda^2-2\lambda\cos\omega+1)}, \ \frac{\lambda^2+2\lambda g-2f}{2a(\lambda^2-2\lambda\cos\omega+1)}\right\}, \quad (\lambda = \lambda_1, \lambda_2, \lambda_3),$$

it being understood that $\lambda_1, \lambda_2, \lambda_3$ are the three *non-zero* roots of the biquadratic in λ, *viz.*,

$$(\lambda^2+2\lambda g-2f)^2 = 4ac(\lambda^2-2\lambda\cos\omega+1).$$

As is to be expected, the first of the four cyclic points found above is no other than the point α. Now simple inspection of the Cartesian equation (10) evidently suggests that the parabola Σ touches the line (4) at its point of intersection with the line (5), so that the actual point of contact of (4) with Σ is simply the point α', *viz.*, the centre of curvature (of Γ) answering to the cyclic point α. Further noticing that the line (4) intersects the axis OY *perpendicularly* at the cyclic point α, and that OY touches the given cubic Γ at α, we conclude that the normal to Γ at α touches Σ at α'.

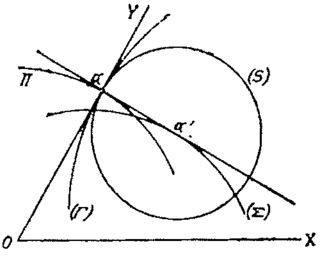

FIG. 3

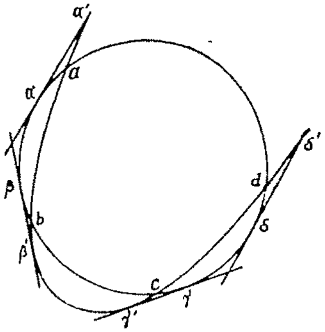

FIG. 4

Besides the same normal (*viz.*, that at α) must touch the evolute Ω (of Γ) at exactly the same point α'.

Thus as could be foreseen from other considerations, *the normal to the (bicursal) circular cubic Γ at a cyclic point α touches the focal parabola Σ as well as the evolute Ω at one and the same point viz., α', (which is the centre of curvature at α).* Fig. 3 shows the relative configuration of the points α, α' and the conic-pair S, Σ. Similar properties must hold likewise for the other three cyclic points β, γ, δ (cognate with α) relatively to their respective centres of curvature β' γ', δ', a fact given prominence to in Fig. 4. Considerations of symmetry at once reveal the existence of analogous relations between the four cognate cyclic points (lying on each of the other three circles Π_1, Π_2, Π_3) and their four centres of curvature lying on the corresponding focal parabola. In this manner without referring at all to the bicircular quartic, (of which every circular cubic can be regarded as a degenerate variety), we can derive an *independent* proof of the following proposition:

The evolute Ω of a bicursal circular cubic Γ possesses 16 cusps (real or imaginary), situated, four by four, on the four focal parabolas Σ, Σ_1, Σ_2, Σ_3. Further Ω touches any of these focal parabolas (say, Σ_r) at each of the four cusps (situated on its periphery), the four common tangents at these points being normal to the cubic Γ at the four cognate cyclic points, that lie on the corresponding circle of inversion Π_r. As a consequence, the 16 cuspidal tangents of Ω consist of the four tetrads of common tangents to the four conic-pairs:

$$(\Pi, \Sigma),\ (\Pi_1, \Sigma_1),\ (\Pi_2, \Sigma_2)\ \textit{and}\ (\Pi_3, \Sigma_3).$$

12. If we proceed on lines similar to those of Art. 10, we can appeal to the equations (6) and (7) of Art 11 and establish therefrom the 'polyzomal' character of a circular cubic Γ. To that end we ascribe three arbitrary values λ_1, λ_2, λ_3 to λ, and express the equation of Γ in each of the three equivalent forms:

$$S_1.y = T_1^2,\quad S_2.y = T_2^2\ \text{and}\ S_3.y = T_3^2, \tag{I}$$

where

$$S_r \equiv S+2\lambda_r x+\lambda_r^2 y\ \text{and}\ T_r = x+\lambda_r y,\quad (r = 1, 2, 3).$$

Palpably the three concurrent lines $\{T_r\}$ conform to the identical relation

$$pT_1+qT_2+rT_3 = 0, \tag{II}$$

where

$$p, q, r \equiv \lambda_2-\lambda_3,\ \lambda_3-\lambda_1,\ \lambda_1-\lambda_2,\ \text{so that}\ p+q+r = 0.$$

In view of (II), the three equations of (I) lead to

$$\sqrt{(lS_1)}+\sqrt{(mS_2)}+\sqrt{(nS_3)} = 0, \tag{III}$$

where l, m, n are constants satisfying the relation

$$\sqrt{l} \pm \sqrt{m} \pm \sqrt{n} = 0.$$

Thus (III) is the desired irrational expression of the circular cubic Γ in terms of three arbitrarily chosen circles of double contact S_1, S_2, S_3.

13. We now propose to scrutinise the inter-relations between four *cognate* cyclic points α, β, γ, δ on a bicursal circular cubic Γ. Evidently the radii of curvature ρ_1, ρ_2, ρ_3, ρ_4 at these points must, as in the case of a bicircular quartic Γ, abide by the relation:

$$\frac{1}{\rho_1} \pm \frac{1}{\rho_2} \pm \frac{1}{\rho_3} \pm \frac{1}{\rho_4} = 0,$$

for the simple reason that Roberts' theorem quoted in Art 4 holds universally for every type of algebraic curve, not excepting a parabola.

Our next query has reference to the polar reciprocal (Ξ) of a focal parabola Σ *w. r. t* the associated circle of inversion Π. Appealing to Fig 4 of Art 11, we gather that the four lines $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$, $\delta\delta'$ being tangents to Σ, their respective poles *viz.*, α, β, γ, δ (taken *w. r. t* the auxiliary circle Π) must lie on the reciprocal curve Ξ; besides, this conic goes also through the centre O of Π, seeing that Σ touches the line at infinity. Thus Ξ passes through all the five points *viz.*, $O, \alpha, \beta, \gamma, \delta$. If we now attend to the patent facts:

(*i*) that the polar conic of the centre of inversion O touches the cubic at O and goes through the cyclic points $\alpha, \beta, \gamma, \delta$, (lying on Π) these being the points of contact of tangents from O to Γ;

and (*ii*) that a conic is *uniquely* defined by five given points (no three of which are collinear),

we cannot escape the conclusion that Ξ is identical with the polar conic of O *w. r. t.* Γ. We may therefore summarise our conclusions in the following garb:

For a bicursal circular cubic Γ, *the polar reciprocal of any of the four focal parabolas with respect to the associated circle of inversion is identical with the polar conic (w. r. t.* Γ) *of the related centre of inversion.*

From this theorem on circular cubics we can, by changing our view-point, deduce immediately an interesting theorem on conics Bearing in mind the elementary principle that, with any *given* parabola Σ as a focal parabola and any *given* circle as the corresponding circle of inversion, a circular cubic is perfectly determinate, having its real asymptote parallel to the directrix of Σ, we are justified in drawing the following inference:

If four lines $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$, $\delta\delta'$ *touch a circle* Π, *(whose centre is* O) *at the points* $\alpha, \beta, \gamma, \delta$ *respectively, then the tangent at* O *to the uniquely determinate conic passing through the five points* $O, \alpha, \beta, \gamma, \delta$ *is parallel to the directrix of the uniquely determinate parabola, that touches the four lines mentioned above.*

In the addendum we shall have occasion to say a few words more about reciprocation with special reference to a bicircular quartic.

ADDENDUM

14. When it is desired to investigate the polar reciprocal (Ξ) of any of the four focal conics (say, Σ) of a bicursal bicircular quartic Γ *w. r. t.* the corresponding circle of inversion Π, we may employ a mode of reasoning almost similar to that of Art. 13. For this purpose we shall still continue to refer to Fig. 4 (of Art. 11), subject to the proviso that Σ is no longer a focal parabola but is rather an ordinary focal conic.

In the first place we notice (as before that the cyclic points $\alpha, \beta, \gamma, \delta$ being the poles *w. r. t.* Π) of the four tangents to Σ, *viz.*, $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$, $\delta\delta'$, must lie on the reciprocal Ξ. The second point to be stressed is that the four cognate foci a, b, c, d (attaching to Π) being its intersections with Σ, their polars *w. r. t.* the (auxiliary circle) Π—which are no other than the tangents to Π at the very same points—must all touch the reciprocal curve Ξ. We are therefore entitled to formulate our conclusions in the following manner:

If, for a given bicursal bicircular quartic Γ, Σ *denote a focal conic and* Π *its associated circle of inversion (having* O *for its centre), and if* U *denote the quadrilateral* *formed by the four 'cognate' cyclic points (seated on* Π) *and V denote the quadrilateral formed by the tangents (drawn to* Π) *at the related tetrad of 'cognate' foci, then there must exist a uniquely determinate conic* Ξ, *which is inscribed in V but circumscribed about U. Furthermore this conic* Ξ *can be identified as the reciprocal of* Σ *w. r. t.* Π. *Finally* Ξ *will pass through the centre of inversion O, when and only when* Γ *reduces to a circular cubic, in this special case,* Ξ *is the same as the polar conic of O w. r. t. the curve.* This proposition loses much of its interest, unless the two sets of points (α, β, γ, δ) and (a, b, c, d) located on Π are all *real*, as visualised in Fig. 4 of Art 9.

15. We shall now close this topic by referring briefly to the *F*-conic* that attaches to the conic-pair (Π, Σ). Remembering that this covariant conic† goes through the eight points of contact (real or imaginary) of Π, Σ, with their tetrad of common tangents, and attending to the results of Arts 4 and 9, we are forthwith led to the following theorem:

For a bicursal bicircular quartic or for a (bicursal) circular cubic, any four cognate cyclic points (situated naturally on a certain circle of inversion Π) *and their attached centres of curvature (situated on the corresponding focal conic* Σ) *make up a set of eight points, lying on a common conic This conic is nothing but the F-conic of* (Π, Σ).

As against this, inquisitive students may propose to investigate the contravariant envelope *viz.*, ϕ-conic—sometimes called the *'harmonic envelope'*—of the conic-pair (Π, Σ).

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* As was first defined by Salmon. the *F*-conic associated with a given conic-pair (S, S') is the locus (of the second degree) of a moving point, the two pairs of tangents drawn from which to S. S' form a harmonic pencil.

† See Askwith's *'Analytical Geometry of Conic Sections'* (1935), Art. 394.

THE DUAL OF A THEOREM PROVED BY F. MORLEY

By

H. F. Baker

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I learn with deep regret, from the March number of the Bulletin of the Calcutta Mathematical Society (Vol. 40, 1948, p. 49) that my old and much respected friend, C. V. Hanumanta Rao, formerly Professor at Lahore, has died. So far as I know his last publications relating to a theorem variously known by the names of Petersen, Hjelmslev and Morley, taken in projective form (Bulletin, 37, 1945, p. 131 and 38 1946, p. 111). I should like to offer the following lines for publication in tribute to his memory. They were actually written in correspondence with him about his last two papers; and, as will be seen, the result might appropriately be called after his name.

In ordinary space of three dimensions, let u, v, w, u', v', w' be six lines, passing through a point T, which lie on a quadric cone. If we take any two ordered triads from these lines, say u, v, w and u', v', w', and consider the six planes

$$\alpha = (v', w), \quad \alpha' = (v, w'), \quad \beta = (w', u), \quad \beta' = (w, u'), \quad \gamma = (u', v), \quad \gamma' = (u, v'),$$

then the three lines of intersection of pairs of these, (α, α'), (β, β'), (γ, γ'), lie in a plane through T. This follows from Pascal's theorem for six points of a conic, and the plane may be called a Pascal plane. We denote it by

$$\omega = \begin{pmatrix} u, & v, & w \\ u', & w', & v' \end{pmatrix}.$$

We consider, with the same notation, also the three other Pascal planes

$$\lambda = \begin{pmatrix} u, & v, & w \\ u', & w', & v' \end{pmatrix}, \qquad \mu = \begin{pmatrix} v, & w, & u \\ v', & u', & w' \end{pmatrix}, \qquad \nu = \begin{pmatrix} w, & u, & v \\ w', & v', & u' \end{pmatrix};$$

by a remark usually ascribed to Steiner, these three planes meet in a line, say s, as will incidentally appear below.

Now take upon the respective lines u, v', w, u', v, w' arbitrary points U, V', W, U', V, W', forming a skew hexagon. There is then, upon the plane λ, a transversal of the opposite sides VW', $V'W$, say l. Equally there is a transversal of the opposite sides WU', $W'U$ of this skew hexagon, lying in the plane μ, say m; and a transversal of the opposite sides UV', $U'V$ lying in the plane ν, say n. In the general case, when the lines l, m, n are skew, there is a quadric surface containing them. We shew that *this quadric surface touches the Pascal plane* ω. For this it is necessary and sufficient that *the points in which the lines l, m, n meet the plane ω should be in line*—and this is what we prove.

Take coordinates x, y, z, t, so that the point of intersection T of the lines $u, v, w,$

u', v', w' is $(0, 0, 0, 1)$, and the plane ω is $x+y+z=0$. Let the planes α, α', which meet on ω, be given by $a_1x+a_2y+a_3z=0$, $a_1'x+a_2'y+a_3'z=0$ so chosen that $a_1'-a_1=1$, $a_2'-a_2=1$, $a_3'-a_2=1$, so that we may write $\alpha'-\alpha=\omega$. Similarly let the planes β, β' be respectively $b_1x+b_2y+b_3z=0$, $b_1'x+b_2'y+b_3'z=0$, with $b_i'-b_i=1$, so that $\beta'-\beta=\omega$; and the planes γ, γ' be respectively $c_1x+c_2y+c_3z=0$, $c_1'x+c_2'y+c_3'z=0$, with $c_i'-c_i=1$, so that $\gamma'-\gamma=\omega$.

The plane λ contains the lines $(w'u, u'v)$, (wu', uv'), or (β, γ), (β', γ'); thus, as $\beta'-\gamma'=\beta-\gamma$, we may write $\lambda=\beta-\gamma$ and $\lambda=\beta'-\gamma'$. Similarly the plane μ contains the lines $(u'v, v'w)$, (uv', vw'), or (γ, α), (γ', α'), and we may write $\mu=\gamma-\alpha$ and $\mu=\gamma'-\alpha'$. And the plane ν contains the lines $(v'w, w'u)$, (vw', wu'), or (α, β), $(\alpha'\ \beta')$, and we may write $\nu=\alpha-\beta$ and $\nu=\alpha'-\beta'$. The planes λ, μ, ν thus meet in a line, s, given by $\alpha=\beta=\gamma$, or $\alpha'=\beta'=\gamma'$.

If (p_1, p_2, p_3) and (q_1, q_2, q_3) be two sets of three numbers, we may denote the three numbers $p_2q_3-p_3q_2$, $p_3q_1-p_1q_3$, $p_1q_2-p_2q_1$, respectively, by $(pq)_1$, $(pq)_2$, $(pq)_3$; so that, if (r_1, r_2, r_3) be another set of three numbers, the sum $(pq)_1r_1+(pq)_2r_2+(pq)_3r_3$ is the determinant (pqr). Then, for instance, the point U on the line u, lies on the planes $(w'u)$, (uv'), or β, γ', and has coordinates

$$[(bc')_1,\ (bc')_2,\ (bc')_3,\ f]$$

where f is a number fixing the position of U on u. And we may, for brevity, denote the four coordinates by $[(bc'), f]$. With this notation the six angular points of the skew hexagon are easily seen to have the respective coordinates

U	V	W	U'	V'	W'
$[(bc'), f]$,	$[(ca'), g]$,	$[(ab'), h]$,	$[(b'c), f']$,	$[(c'a), g']$,	$[(a'b), h']$

with suitable values for the six numbers f, g, h, f', g', h'. Using then U, V, $\cdots$ for these sets of coordinates, we can at once prove that the points $U'-U$, $V'-V$, $W'-W$ lie on the Pascal plane ω, being the points where the diagonals of the skew hexagon $UV'WU'VW'$ meet this plane. For instance, the condition that the point $U'-U$, whose coordinates are

$$[(b'c)_1-(bc')_1,\ (b'c)_2-(bc')_2,\ (b'c)_3-(bc')_3,\ f'-f],$$

should be in the plane $x+y+z=0$, is $(b'c1)-(bc'1)=0$, which is verified in virtue of $b_i'-b_i=1$, $c_i'-c_i=1$.

Also, we have seen that the plane λ, or

$$\begin{pmatrix} u, & v, & w \\ u', & w', & v' \end{pmatrix},$$

is given by $\beta-\gamma=0$, or $\beta'-\gamma'=0$. From this we can at once prove that this plane meets the two opposite sides VW', $V'W$ of the skew hexagon in the respective points $V+W'$, $V'+W$. For the first point, $V+W'$, has coordinates $[(ca')+(a'b),\ g+h']$, and the condition thus lies on $\beta-\gamma=0$, is

$$(ca'b)+(a'bb)-(ca'c)-(a'bc)=0$$

which is satisfied; and $V'+W$ likewise lies on this plane, namely, $[(c'a)+(ab'),\ g'+h]$ lies on $\beta'-\gamma'=0$ because

$$(c'ab')+(ab'b')-(c'ac')-(ab'c') = 0.$$

The transversal of the sides VW', $V'W$ which lies in the plane λ, thus contains the point $(V'+W)-(V+W')$, or $V'-V-(W'-W)$; and this point lies on the plane ω, which, we have seen, is determined by the three points $U'-U$, $V'-V$, $W'-W$. Similarly for the two other pairs of opposite sides of the hexagon, namely, WU', $W'U$ and UV', $U'V$. The transversals of the three pairs of opposite sides of the skew hexagon lying respectively in the planes λ, μ, ν, thus meet the plane ω in the respective points

$$V'-V-(W'-W), \quad W'-W-(U'-U), \quad U'-U-(V'-V);$$

and these are in line. *So that the enunciated theorem is proved.*

If we take the dual of the figure considered, we shall have, in the plane which is the dual of the point T, six lines $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}'$, $\mathbf{v}'$, $\mathbf{w}'$, touching a conic, and six points

$$A = (\mathbf{v}', \mathbf{w}), \quad A' = (\mathbf{v}, \mathbf{w}'), \quad B = (\mathbf{w}', \mathbf{u}), \quad B' = (\mathbf{w}, \mathbf{u}'), \quad C = (\mathbf{u}', \mathbf{v}), \quad C' = (\mathbf{u}, \mathbf{v}'),$$

respective duals of the plane α, α', β, β', γ, γ', forming two triangles ABC, $A'B'C'$, which, by Brianchon's theorem, are in perspective, from a point O the dual of the plane ω, namely, this centre of perspective O is the dual of the Pascal plane

$$\begin{pmatrix} u, & v, & w \\ u', & v', & w' \end{pmatrix}.$$

Conversely two such triangles ABC, $A'B'C'$ in perspective give rise to six lines touching a conic.

The Pascal plane λ, containing the lines $(uw', u'v)$, $(uv', u'w)$, has for dual a point L, through which pass the two lines joining the pairs of points $(\mathbf{w}'\mathbf{u}, \mathbf{u}'\mathbf{v})$, $(\mathbf{u}\mathbf{v}', \mathbf{u}'\mathbf{w})$, namely the lines BC, $B'C'$, which are corresponding sides of the two triangles ABC, $A'B'C'$. Likewise M, the dual point of the plane μ, is the intersection of CA, $C'A'$; and N, the dual of ν, is the intersection of AB, $A'B'$. Thus the Steiner remark that the Pascal planes λ, μ, ν meet in a line becomes in the dual figure the fact that the perspective triangles have an axis of perspectivity, LMN.

The six sides $V'W$, WU', . . . of the skew hexagon in the original figure are similarly seen to give rise, in the dual figure, to lines a, b', . . ., drawn respectively through the points A, B', . . ., which form a skew hexagon in the dual figure. And the transversal of VW', $V'W$ in the plane λ, gives rise to the transversal of the opposite sides a, a', drawn from L. This transversal, and the two others from M to b, b', and from N to c, c', are thus all met by a line in the dual figure passing through the point O, the centre of perspective of the two triangles. And *this is the projective form of the theorem, known as Morley's, referred to.*

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A NOTE ON HAMY'S THEOREM

By

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M. Hamy (1889) discovered the theorem that a heterogeneous liquid rotating in relative equilibrium about an axis cannot have an ellipsoidal density-stratification. He proved it from the dynamical equations,

$$\omega^2 x = -\frac{\partial\Phi}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{1}$$

$$\omega^2 y = -\frac{\partial\Phi}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} \tag{2}$$

$$0 = -\frac{\partial\Phi}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} \tag{3}$$

where ω is the constant angular velocity about the z-axis, Φ the gravitational potential, ρ the density and p the hydrostatic pressure at the point x, y, z, by utilising the well-known form for the gravitational potential of a homogeneous ellipsoidal shell. The purpose of the present note is to show that the theorem is a consequence of the consistency of the above dynamical equations with the gravitational equation

$$\nabla^2\Phi \equiv \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = -4\pi G\rho \tag{4}$$

and its validity does not really depend on the particular form of the potential of an ellipsoidal shell. To prove this we proceed as follows: From equations (1), (2), (3)

$$\omega^2(x\,dx + y\,dy) = -d\Phi + \frac{dp}{\rho} \tag{5}$$

In (5) $\omega^2(x\,dx+y\,dy)$ is a perfect differential, ω being a constant and so is $d\Phi$. Hence dp/ρ must be a perfect differential in order that integration of (5) may be possible. Thus we must have

$$\int\frac{dp}{\rho} = f(\rho) \tag{6}$$

unless ρ is constant, and hence

$$p = \int f'(\rho)\rho\,d\rho = P(\rho), \text{ say,} \tag{7}$$

From (5) we have

$$\omega^2\frac{(x^2+y^2)}{2} = -\Phi + f(\rho) \tag{8}$$

Operating with ∇^2 on both sides and using (4) we have

$$2\omega^2 = 4\pi G\rho + \nabla^2 f(\rho) \tag{9}$$

Now let

$$\rho = \psi(\alpha x^2 + \beta y^2 + \gamma z^2) \tag{10}$$

Substituting in (9) and putting

$$f(\rho) = f\{\psi(\alpha x^2+\beta y^2+\gamma z^2)\} \equiv F(\alpha x^2+\beta y^2+\gamma z^2) \qquad (11)$$

we have,

$$2\omega^2 = 4\pi G\rho + \nabla^2 F(\alpha x^2+\beta y^2+\gamma z^2).$$

That is,

$$4(\alpha^2x^2+\beta^2y^2+\gamma^2z^2)F'' + 2(\alpha+\beta+\gamma)F' + 4\pi G\rho = 2\omega^2 \qquad (12)$$

where F' and F'' denote differentiations with respect to the argument $\alpha x^2+\beta y^2+\gamma z^2$.

(12) reduces to

$$2(\alpha^2x^2+\beta^2y^2+\gamma^2z^2)F'' + (\alpha+\beta+\gamma)F' + 2\pi G.\psi(\alpha x^2+\beta y^2+\gamma z^2) = \omega^2. \qquad (13)$$

If (13) is to be integrable (as must happen in any possible case) we must have, either

$$F''=0, \quad \text{giving} \quad F'=\text{const.}$$

and hence from (13)

$$\psi(\alpha x^2+\beta y^2+\gamma z^2) = \rho = \text{const.};$$

or, $\alpha^2x^2+\beta^2y^2+\gamma^2z^2$ must be a function of $\alpha x^2+\beta y^2+\gamma z^2$ which is impossible unless

$$\alpha = \beta = \gamma.$$

That is, if the fluid be heterogeneous, no strictly ellipsoidal distribution of density is possible, which proves the theorem.

It must be noted here that though the original theorem only mentions a heterogeneous liquid, it applies nevertheless to any heterogeneous fluid, including a gas by virtue of equation (6).

There are two points to be observed in connection with the above proof of Hamy's theorem.

There is no assumption with regard to the form of the boundary of the rotating fluid at the start. And so, no use has been made of a definite form of the potential function. The mere consistancy of the dynamical equations and Poisson's gravitational equation excludes the possibility of ellipsoidal stratification and so generally of ellipsoidal boundaries. The case $\rho = \text{const.}$ is a special one, as stated before. The method may be useful in eliminating some other forms of stratifications.

In equations 1—4, Φ may represent the self-gravitation of the rotating fluid together with the potential due to external gravitating bodies (which would not disturb the conditions of motion assumed). The dynamical problem will not thereby be changed, and the above conclusion will remain valid. Hence Hamy's theorem will not only be true under the self-gravitation of a rotating fluid mass but also when the same is placed in an external gravitational field which would not disturb the character of the motion.

In conclusion I must express my grateful thanks to Prof. N. R. Sen for his kind interest and encouragement.

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